GENUS 2 FIELDS WITH DEGREE 3 ELLIPTIC SUBFIELDS

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Abstract. In this paper we study genus 2 function fields K with degree 3 elliptic subfields. We compute an equation for the locus of these fields K in the moduli space of genus 2 curves. From this we derive that the number of $Aut(K)$-classes of such subfields of fixed K is 0,1,2 or 4.

1. Introduction

We study genus two curves $\mathcal{C}$ whose function fields have a degree 3 subfield of genus 1. Such subfields we call elliptic subfields. Such curves $\mathcal{C}$ have already occurred in the work of Hermite, Goursat, Burkhardt, Brioschi, and Bolza, see Krazer [6] (p. 479). More generally, degree n elliptic subfields of genus 2 fields have been studied by Frey [2], Frey and Kani [3], Kuhn [8], Shaska [11], [13], and Shaska and Voelklein [14]. In the degree 3 case, explicit equations can be used to answer questions which remain open in the general case. This was the theme of the author's PhD thesis (see [12]) which also covered the case of degree 2 (which is subsumed in [14]).

Equation (5) gives a normal form for pairs (K, E) where E is a degree 3 elliptic subfield of K. This normal form depends on two parameters $a, b \in k$. Isomorphism classes of pairs (K, E) are parameterized by parameters u, v where $u := ab$ and $v := b^3$. This yields an expression for the invariants (i_1, i_2, i_3) of K (see Igusa [9]) in terms of u and v. Our central object of study is the map

$$\theta : (u, v) \to (i_1, i_2, i_3)$$

From the general theory (see above) it is clear that this map has degree 2, which means that a general genus 2 field having elliptic subfields of degree 3 has exactly two such subfields. From the results of this paper we quickly derive that there are exactly four genus 2 fields which have four $Aut(K)$-classes of such subfields, and a 1-dimensional family with exactly one $Aut(K)$-class of such subfields.

The main goal of this paper is to compute an equation for the locus $\mathcal{L}_3$ of genus 2 fields with elliptic subfields of degree 3. This equation is in terms of the Igusa coordinates i_1, i_2, i_3 on the moduli space of genus 2 curves. It is displayed as equation (33) in Appendix A (because of its length). The direct approach to finding this equation would be to eliminate u and v from the explicit equations defining θ (see (13)). However, this direct approach exceeds available computer power, so we introduce auxiliary parameters r_1, r_2 that parameterize pairs of cubic polynomials. They arise from the fact that the subfield E induces a particular sextic defining K which splits naturally as a product of 2 cubic polynomials. We show that θ factorizes as

$$(u, v) \to (r_1, r_2) \to (i_1, i_2, i_3)$$

where the latter map is birational, and the former has degree 2. Thus r_1, r_2 yield a birational parameterization of the locus $\mathcal{L}_3$ of genus 2 fields having a degree 3 elliptic subfield. That $\mathcal{L}_3$ is a rational variety follows also from the general theory of "diagonal modular surfaces", see Kani [5].

The formulas expressing i_1, i_2, i_3 in terms of r_1, r_2 are much smaller than those in terms of u and v. Now we can eliminate r_1 and r_2 to obtain the desired equation (33). We further find relations between the j-invariants of the degree 3 elliptic subfields of K and classify all K in $\mathcal{L}_3$ with extra automorphisms.

All the computations were done using Maple [10].

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2. Genus Two Fields With Degree 3 Elliptic Subfields

Let k be an algebraically closed field of characteristic 0. All function fields will be over k. Let K be a genus 2 function field and E a degree 3 elliptic subfield of K.

The associated extension $k(X)/k(U)$. It is well-known that K has exactly one genus zero subfield of degree 2, which we denote by $k(X)$. The generator of $Gal(K/k(X))$ is the hyperelliptic involution of K. It fixes each elliptic subfield of K, see Tamme [15]. Hence the field $E \cap k(X)$, denote it by $k(U)$, is a subfield of E of degree 2.

$$\begin{array}{ccc} K & \overset{2}{-} & k(X) \\ 3\,| & & |\,3 \\ E & \overset{2}{-} & k(U) \end{array}$$

Ramification of K/E. Either K/E is ramified at exactly 2 places of K, of ramification index 2, or at one place of K, of ramification index 3. It follows immediately from the Riemann-Hurwitz formula. The former (resp., latter) case we call the non-degenerate (resp., degenerate) case, as in [2].

Definition 1. A **non-degenerate pair** (resp., **degenerate pair**) is a pair (K, E) such that K is a genus 2 field with a degree 3 elliptic subfield E where the extension K/E is ramified in two (resp., one) places. Two such pairs (K, E) and (K', E') are called isomorphic if there is a k-isomorphism $K \to K'$ mapping $E \to E'$.

Ramification of $k(X)/k(U)$. In the non-degenerate (resp., degenerate) case $k(X)/k(U)$ is ramified at exactly 4 (resp., 3) places of $k(X)$ each of ramification index 2 (resp., one of index 3 and the other two of index 2), see [2].

Invariants of K. We denote by J_2, J_4, J_6, J_{10} the classical invariants of K, for their definitions see [9] or [14]. These are homogeneous polynomials (of degree indicated by subscript) in the coefficients of a sectix $f(X, Z)$ defining K

$$Y^2 = f(X, Z) = a_6 X^6 + a_5 X^5 Z + \cdots + a_1 X Z^5 + a_0$$

and they are a complete set of $SL_2(k)$-invariants (acting by coordinate change). The absolute invariants

$$i_1 := 144\frac{J_4}{J_2^2}, \quad i_2 := -1728\frac{J_2 J_4 - 3J_6}{J_2^3}, \quad i_3 := 486\frac{J_{10}}{J_2^5} \tag{1}$$

are even $GL_2(k)$-invariants. Two genus 2 curves with $J_2 \neq 0$ are isomorphic if and only if they have the same absolute invariants.

Main Theorem: *Let K be a genus 2 field and $e_3(K)$ the number of $Aut(K/k)$-classes of elliptic subfields of K of degree 3. Then;*

i) $e_3(K) = 0, 1, 2,$ or 4

ii) $e_3(K) \geq 1$ if and only if the classical invariants of K satisfy the irreducible equation $F(J_2, J_4, J_6, J_{10}) = 0$ displayed in Appendix A.

There are exactly two genus 2 curves (up to isomorphism) with $e_3(K) = 4$, see 4.2. The case $e_3(K) = 1$ (resp., 2) occurs for a 1-dimensional (resp., 2-dimensional) family of genus 2 curves, see section 4.

2.1. **The non-degenerate case.** Let (K, E) be a non-degenerate pair and $k(X)$ and $k(U)$ their associated genus 0 subfields. Both $k(X)/k(U)$ and $E/k(U)$ are ramified at 4 places of $k(U)$, three of which are in common. Take the common places to be $U = q_1, q_2, q_3$. Also, take $U = 0$ (resp., $U = \infty$) the place ramified in $k(X)$ but not in E (resp., in E but not $k(X)$). Take $X = 0$ (resp., $X = \infty$) the place over $U = 0$ of ramification index 2 (resp., 1). In the following figure bullets (resp., circles) represent places of ramification index 2 (resp., 1).

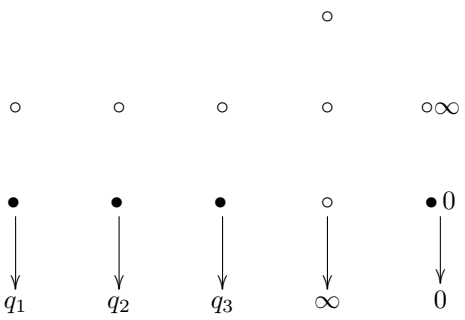


FIGURE 1. Ramification of $k(X)/k(U)$

Thus, $U = \phi(X) = l\frac{X^2}{X^3+aX^2+bX+c}$, where $l, a, b, c \in k$ such that $X^3 + aX^2 + bX + c$ has no multiple roots and $l, c \neq 0$. We can make $l = 1$ and $c = 1$ by replacing U and X by scalar multiples. Then,

$$U = \phi(X) = \frac{X^2}{X^3 + aX^2 + bX + 1} \tag{2}$$

Remark 2. *These conditions determine X up to multiplication by a third root of unity $\xi_3 \in k$. Replacing X by $\xi_3 X$ replaces (a, b) by $(\xi_3^2 a, \xi_3 b)$. Invariants of this transformation are:*

$$\begin{aligned} u &= ab \\ v &= b^3 \end{aligned} \tag{3}$$

The derivative of $\phi(X)$ is

$$\phi'(X) = -\frac{X(X^3 - bX - 2)}{(X^3 + aX^2 + bX + 1)^2}$$

Taking the resultant with respect to z of $z^2 - bz - 2$ and $\frac{\phi(x)-\phi(z)}{x-z}$ we get

$$(x^3 - bx - 2)(4x^3 + b^2x^2 + 2bx + 1) \tag{4}$$

The roots of this polynomial correspond to the 6 places of $k(X)$ over the places $U = q_1, q_2, q_3$. Thus, the 6 places of $k(X)$ ramified in K correspond to the roots of the polynomial on the right side of

$$Y^2 = (X^3 + aX^2 + bX + 1)\,(4X^3 + b^2X^2 + 2bX + 1) \tag{5}$$

which gives an equation of K. The discriminant of the sextic is nonzero, hence

$$\Delta := (4a^3 + 27 - 18ab - a^2b^2 + 4b^3)^2(b^3 - 27) \neq 0 \tag{6}$$

One checks that the element

$$V = Y\,\frac{X^3 - bX - 2}{F(X)^2} \tag{7}$$

satisfies

$$V^2 = (U - q_1)(U - q_2)(U - q_3) = U^3 + 2\frac{ab^2 - 6a^2 + 9b}{R}\,U^2 + \frac{12a - b^2}{R}\,U - \frac{4}{R}$$

where $R := \frac{\Delta}{(b^3-27)}$. Thus, $E = k(U, V)$.

Lemma 3. *For $(a, b) \in k^2$ with $\Delta \neq 0$, equation (5) defines a genus 2 field $K_{a,b} = k(X, Y)$. It has a non-degenerate degree 3 elliptic subfield $E_{a,b} = k(U, V)$, where U and V are given in (2) and (7). Two such pairs $(K_{a,b}, E_{a,b})$ and $(K_{a',b'}, E_{a',b'})$ are isomorphic if and only if $u = u'$ and $v = v'$ (where u, v and u', v' are associated with a, b and a', b', respectively, by (3)).*

Proof. The first two statements follow by reversing the above arguments. If $u = u'$, $v = v'$ then $a' = \xi_3^i a$, $b' = \xi_3^i b$ for some i. Then clearly the two associated non-degenerate pairs are isomorphic. The converse follows from Remark 2.

□

Proposition 4. *The $(u, v) \in k^2$ with $\Delta \neq 0$ bijectively parameterize the isomorphism classes of non-degenerate pairs (K, E) (via the parameterization defined in Lemma 3).*

Proof. The proof follows from Lemma 3.

□

From the normal form of K in (5) we can compute the classical invariants J_2, J_4, J_6, J_{10} in terms of u, v. These expressions satisfy equation (33) which proves one implication of claim ii) of the theorem for the non-degenerate pairs. In section 4 we explain how equation (33) was found.

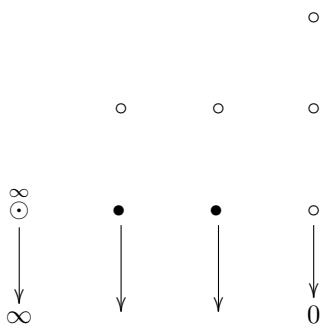


FIGURE 2. Ramification of $k(X)/k(U)$, degenerate case.

2.2. **The degenerate case.** Let K/E be a degenerate pair. Then $k(X)/k(U)$ is ramified at exactly three places of $k(U)$ which are also ramified in E. Let $U = 0$ be the place ramified in E but not in $k(X)$. Take $U = \infty$ be the place of $k(U)$ that is totally ramified in $k(X)$ and $X = \infty$ the place over it.

Then, $U = l(X^3 + aX^2 + bX + c)$ for $a, b, c, l \in k$ such that $X^3 + aX^2 + bX + c$ has no multiple roots and $l \neq 0$. Replacing X by $X + \frac{a}{3}$ and U by a scalar multiple we get $U = X^3 + bX + c$. Note that $b \neq 0$ (otherwise two of the places of $k(U)$ ramified in $k(X)$ coalesce). Thus, by replacing X by a scalar multiple we can also get $b = 1$. Then,

$$U = X^3 + X + c \tag{8}$$

As in Lemma 3 we get $K = k(X, Y)$ such that

$$Y^2 = (3X^2 + 4)(X^3 + X + c) \tag{9}$$

where $c^2 \neq -\frac{4}{27}$. Then, $V = Y(3X^2 + 1)$ satisfies the equation

$$V^2 = U(27U^2 - 54cU + 4 + 27c^2)$$

Thus, $E = k(U, V)$.

Remark 5. *In this case, those $w := c^2$ with $w \neq -\frac{4}{27}$ bijectively parameterize the isomorphism classes of degenerate pairs K/E. The proof is analogous to the non-degenerate case.*

Since

$$\begin{aligned} J_2 &= 774 \\ J_4 &= 36(268 + 837w) \\ J_6 &= 36(76760 + 290574w - 729w^2) \\ J_{10} &= 432(27w + 4) \end{aligned} \tag{10}$$

the map

$$\begin{aligned} k \setminus \{-\frac{4}{27}\} &\to k^3 \\ w &\to (i_1, i_2, i_3) \end{aligned}$$

is injective. Thus, a genus 2 field K has at most one $Aut(K)$-orbit of subfields E such that (K, E) is a degenerate pair. One can check that J_2, J_4, J_6, J_{10} satisfy

equation (33). This completes the proof that the condition in part (ii) of Main Theorem is necessary.

From (10) we compute that the locus of those K being part of a degenerate pair (K, E) is given by

$$\begin{aligned} & 3418801i_1^2 - 2550732480i_1 + 4023336960 + 611249816i_2 = 0 \\ & 161446939560824832i_3 + 23630752512i_1^2 - 6321363049i_1^3 - 29445783552i_1 \\ & + 12230590464 = 0 \end{aligned} \tag{11}$$

(Note $J_2 = 774 \neq 0$ on this locus, so i_1, i_2, i_3 are everywhere defined). For later use we note the following: If (K, E_0) is a non-degenerate pair with parameters u, v then there exists a degenerate pair (K, E_1) (for same K) if and only if

$$2v - 9u + 27 = 0 \tag{12}$$

This is obtained by expressing i_1, i_2, i_3 in (11) in terms of u, v.

3. Function Field of $\mathcal{L}_3$

The absolute invariants i_1, i_2, i_3 in terms of u, v are

(13)

$$\begin{aligned} i_1 =& \frac{144}{v(-405 + 252u + 4u^2 - 54v - 12uv + 3v^2)^2}(1188u^3 - 8424uv + u^4v - 24u^4 \\ & + 14580v - 66u^3v + 138uv^2 + 297u^2v + 945v^2 - 36v^3 + 9u^2v^2) \\ i_2 =& -\frac{864}{v^2(-405 + 252u + 4u^2 - 54v - 12uv + 3v^2)^3}(-81v^3u^4 + 2u^6v^2 + 234u^5v^2 \\ & + 3162402uv^2 - 21384v^3u + 26676v^4 - 473121v^3 - 72u^6v - 5832v^4u + 14850v^3u^2 \\ & - 72v^3u^3 + 324v^4u^2 - 650268u^3v - 5940u^3v^2 - 3346110v^2 + 432u^6 - 1350u^4v^2 \\ & + 136080u^4v - 7020u^5v - 307638u^2v^2 \\ i_3 =& -243\frac{(v-27)(4u^3 - u^2v - 18uv + 4v^2 + 27v)^3}{v^3(-405 + 252u + 4u^2 - 54v - 12uv + 3v^2)^5} \end{aligned}$$

We will sometimes view u, v as parameters from k and sometimes as the corresponding coordinate functions on k^2. From the context it will be clear which point of view we are taking. If we could eliminate u, v from equations (13) then the desired equation (33) would be obtained directly. However, this exceeds available computer power. We use additional invariants r_1, r_2 to overcome this problem in the next section.

As mentioned in the Introduction, the general theory of [2] and [8] implies that $[k(i_1, i_2, i_3) : k(u, v)] = 2$. We also obtain this as a by-product of our computations and the following Lemma, which we record for completeness.

Lemma 6. $[k(i_1, i_2, i_3) : k(u, v)] \leq 2$

Proof. From the resultants of equations in (13) we determine that $[k(v) : k(i_1, i_2)] = 16$, $[k(v) : k(i_2, i_3)] = 40$, and $[k(v) : k(i_1, i_3)] = 26$. We can show that $u \in k(i_1, i_2, i_3, v)$, the expression is large and we don't display it (the interested reader can check [12]). Since $[k(u, v) : k(i_1, i_2, i_3)]$ must be a common divisor of 16, 26, and 40, then the claim follows.

□

3.1. **Invariants of Two Cubics.** We define the following invariants of two cubic polynomials. For $F(X) = a_3X^3 + a_2X^2 + a_1X + a_0$ and $G(X) = b_3X^3 + b_2X^2 + b_1X + b_0$ define

$$H(F,G) := a_3b_0 - \frac{1}{3}a_2b_1 + \frac{1}{3}a_1b_2 - a_0b_3$$

We denote by $R(F,G)$ the resultant of F and G and by $D(F)$ the discriminant of F. Also,

$$r_1(F,G) = \frac{H(F,G)^3}{R(F,G)}, \quad r_2(F,G) = \frac{H(F,G)^4}{D(F)\,D(G)}$$

Remark 7. *In* [7] *it is shown that* r_1, r_2*, and* $r_3 = \frac{H(F,G)^2}{J_2(F\,G)}$ *form a complete system of invariants for unordered pairs of cubics.*

For $F(X) = X^3 + aX^2 + bX + 1$ and $G(X) = 4X^3 + b^2X^2 + 2bX + 1$ and u, v as in (3) we have

$$\begin{aligned} r_1(F,G) &= 27\frac{v(v-9-2u)^3}{4v^2 - 18uv + 27v - u^2v + 4u^3} \\ r_2(F,G) &= -1296\frac{v(v-9-2u)^4}{(v-27)(4v^2 - 18uv + 27v - u^2v + 4u^3)} \end{aligned} \tag{14}$$

Taking the resultants from the above equations we get the following equations for u and v over $k(r_1, r_2)$:

$$\begin{aligned} &65536r_1r_2^3\,u^2 + (42467328r_2^4 + 21233664r_2^4r_1 + 480r_2r_1^4 + 2r_1^5 + 41472r_2^2r_1^3 \\ &+1548288r_2^3r_1^2 - 294912r_2^3r_1)u - 382205952r_2^4 + 238878720r_2^4r_1 - 2654208r_2^3r_1 \\ &+13934592r_2^3r_1^2 + 285696r_2^2r_1^3 + 2400r_2r_1^4 + 7r_1^5 = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} &16384v^2r_2^3 + (221184r_2^3r_1 + r_1^4 + 11520r_2^2r_1^2 - 442368r_2^3 + 192r_2r_1^3)v \\ &-5971968r_2^3r_1 - 864r_2r_1^3 - 124416r_2^2r_1^2 - 2r_1^4 = 0 \end{aligned} \tag{16}$$

Roots of equation (15) (resp., equation (16)) are u and $\beta(u)$ (resp., v and $\beta(v)$) where,

$$\begin{aligned} \beta(u) &= \frac{(v-3u)(324u^2 + 15u^2v - 378uv - 4uv^2 + 243v + 72v^2)}{(v-27)(4u^3 + 27v - 18uv - u^2v + 4v^2)} \\ \beta(v) &= -\frac{4(v-3u)^3}{4u^3 + 27v - 18uv - u^2v + 4v^2} \end{aligned} \tag{17}$$

It follows that either $[k(u,v) : k(r_1,r_2)] = 2$ or $k(u,v)/k(r_1,r_2)$ is Galois with Klein 4-group as Galois group. In the latter case, the map

$$\begin{aligned} \tau : k(u,v) &\to k(u,v) \\ u &\to u \\ v &\to \beta(v) \end{aligned} \tag{18}$$

is an involutory automorphism of $k(u,v)$. But by plugging in equation (17) we see that it is not involutory. Thus, $[k(u,v) : k(r_1,r_2)] = 2$ and $Gal_{k(u,v)/k(r_1,r_2)} = \langle\beta\rangle$.

Lemma 8. *The fields $k(i_1,i_2,i_3)$ and $k(r_1,r_2)$ coincide, hence $[k(u,v):k(i_1,i_2,i_3)]=2$. Moreover;*

(19)

$$i_1=\frac{9}{4}\frac{(13824r_1^3r_2^2+442368r_1^2r_2^3+5308416r_1r_2^4+192r_1^4r_2+r_1^5+786432r_1r_2^3+9437184r_2^4)}{r_1(-1152r_2^2+96r_2r_1+r_1^2)^2}$$

$$\begin{aligned}i_2=&\frac{27}{8r_1^2(-1152r_2^2+96r_2r_1+r_1^2)^3}(+79626240r_1^4r_2^4-4076863488r_1^2r_2^5+34560r_1^6r_2^2\\&+12230590464r_1^2r_2^6+32614907904r_1r_2^6+14495514624r_2^6+288r_1^7r_2+2211840r_1^5r_2^3\\&+r_1^8-212336640r_1^3r_2^4+1528823808r_1^3r_2^5-2359296r_1^4r_2^3)\end{aligned}$$

$$i_3=-521838526464\frac{r_2^9}{r_1^2(-1152r_2^2+96r_2r_1+r_1^2)^5}$$

Proof. Equations (19) are verified by expressing all variables in terms of u,v (These equations were found by taking suitable resultants). It follows that $k(i_1,i_2,i_3)\subset k(r_1,r_2)$. Equality follows since $[k(u,v):k(r_1,r_2)]=2$ and $[k(u,v):k(i_1,i_2,i_3)]\leq 2$ (see Lemma 6).

□

Finding equation (33): To find equation (33) we eliminate r_1 and r_2 from equations (19). The resulting equation (33) has degree 8, 13, and 20 in i_1,i_2,i_3 respectively.

4. Completing the proof of the Main Theorem

The map

$$\theta:(u,v)\to(i_1,i_2,i_3)$$

given by (13) has degree 2 and it is defined when $J_2\neq 0$. For now we assume that $J_2\neq 0$ (The case $J_2=0$ is treated in section 4.2). Denote the minors of the Jacobian matrix of θ by $M_1(u,v),M_2(u,v),M_3(u,v)$. The solutions of

$$\begin{cases}M_1(u,v)=0\\M_2(u,v)=0\\M_3(u,v)=0\end{cases}\tag{20}$$

consist of the (non-singular) curve

$$8v^3+27v^2-54uv^2-u^2v^2+108u^2v+4u^3v-108u^3=0\tag{21}$$

and 7 isolated solutions which we display in table (1) together with the corresponding values (i_1,i_2,i_3) and properties of the corresponding genus 2 field K.

Let $\bar{u},\bar{v},\bar{i}_1,\bar{i}_2,\bar{i}_3$ denote the restrictions of the corresponding functions to the curve (21). Eliminating u from (21) and the defining equation for i_1 in (13) we get a relation between v and i_1 which shows $[k(\bar{v},\bar{i}_1):k(\bar{i}_1)]=8$; also we get $\bar{u}$ expressed as a rational function of $\bar{v}$ and $\bar{i}_1$. Hence $k(\bar{v},\bar{i}_1)=k(\bar{u},\bar{v})$ and so $[k(\bar{u},\bar{v}):k(\bar{i}_1)]=8$. Similarly we get a relation between $\bar{v}$ and $\bar{i}_2$. Eliminating $\bar{v}$ gives

$$G(\bar{i}_1,\bar{i}_2)=0\tag{22}$$

an irreducible equation between $\bar{i}_1$ and $\bar{i}_2$ of degree 8 (resp., 12) in $\bar{i}_1$ (resp., $\bar{i}_2$). It is fully displayed as equation (34) in Appendix B. Thus, $[k(\bar{i}_1,\bar{i}_2):k(\bar{i}_1)]=8$. Since also $[k(\bar{u},\bar{v}):k(\bar{i}_1)]=8$, it follows that $k(\bar{u},\bar{v})=k(\bar{i}_1,\bar{i}_2)$. Hence, $e_3(K)=1$

(u,v)	(i_1,i_2,i_3)	$Aut(K)$	$e_3(K)$
$(-\frac{7}{2},2)$	$J_{10}=0$, no associated genus 2 field K		
$(-\frac{775}{8},\frac{125}{96})$, $(\frac{25}{2},\frac{250}{9})$	$-\frac{8019}{20},-\frac{1240029}{200},\frac{531441}{100000}$	D_4	2
$(27-\frac{77}{2}\sqrt{-1},23+\frac{77}{9}\sqrt{-1})$, $(27+\frac{77}{2}\sqrt{-1},23-\frac{77}{9}\sqrt{-1})$	$(\frac{729}{2116},\frac{1240029}{97336},\frac{531441}{13181630464}$	D_4	2
$(-15+\frac{35}{8}\sqrt{5},\frac{25}{2}+\frac{35}{6}\sqrt{5})$, $(-15-\frac{35}{8}\sqrt{5},\frac{25}{2}-\frac{35}{6}\sqrt{5})$	$81,-\frac{5103}{25},-\frac{729}{12500}$	D_6	2

TABLE 1. Exceptional points where $det(Jac(\theta))=0$

for any K such that the associated u and v satisfy equation (21) except possibly those (u,v) that lie over singular points of the curve (22). We check that all these singular points have multiplicity 2, hence there are at most two (u,v) points over each of them.

Remark 9. *If K has one subfield of degree 3, then it has at least two distinct such subfields. This is actually true for degree n elliptic subfields, for any n, see [2] and [8]). (For $n=3$ it also follows from the explicit construction in* [12]*, Lemma 5.4.) Since $e_3(K)=1$ for a generic curve K of the locus* (22)*, such K has $|Aut(K)|>2$. Actually $Aut(K)=V_4$. We verified this by using equation (17) from* [14] *which gives a necessary and sufficient condition for a genus 2 field K to have $V_4\leq Aut(K)$ (in terms of the classical invariants of K). Taking the resultant of this equation and the one in Appendix A yields an expression that has* (22) *(the projectivized version) as a factor. From the conditions given in* [14] *for K to have $Aut(K)$ strictly bigger than V_4, one checks that actually $Aut(K)=V_4$ for all K in locus* (22)*.*

4.1. **The general case.** By previous section θ is generically a covering of degree 2. So there exists a Zariski open subset $\mathcal{U}$ of k^2 with the following properties: Firstly, θ is defined everywhere on $\mathcal{U}$ and is a covering of degree 2 from $\mathcal{U}$ to $\theta(\mathcal{U})$. Further, if $\mathfrak{u}\in\mathcal{U}$ then all $\mathfrak{u}'\in k^2$ with θ defined at $\mathfrak{u}'$ and $\theta(\mathfrak{u}')=\theta(\mathfrak{u})$ also lie in $\mathcal{U}$. Suppose $\underline{i}\in k^3$ such that $|\theta^{-1}(\underline{i})|>2$ and $det(Jac(\theta))$ does not vanish at any point of $\theta^{-1}(\underline{i})$. Then by implicit function theorem, there is an open ball B around each element of $\theta^{-1}(\underline{i})$ such that each point in $\theta(B)$ has >2 inverse images under θ. But B has to intersect the Zariski open set $\mathcal{U}$. This is a contradiction. Thus, if $\underline{i}\in k^3$ and $|\theta^{-1}(\underline{i})|>2$, then $det(Jac(\theta))=0$ at some point of $\theta^{-1}(\underline{i})$ and so $\underline{i}$ satisfies (22). It follows that if K is a genus 2 field with $J_2\neq 0$ and $G\neq 0$ then $e_3(K)=2$.

Note that if there is a degenerate pair (K,E_1) and a non-degenerate pair (K,E_2) (for the same K) then for the (u,v)-invariants of the latter we have (12), hence $\beta(v)=27$ and so the β-image of (u,v) doesn't correspond to a non-degenerate pair.

We can now complete the proof of part (ii) of Main Theorem. We noted before that the condition in (ii) is necessary. This condition (see Appendix A) defines an irreducible sublocus of $\mathcal{M}_2$ because the corresponding equation is irreducible (by Maple). The locus $e_3(K)\geq 1$ lies in this sublocus and contain a dense subset of it

(image of θ). By Lange [4], the locus $e_3(K) \geq 1$ is Zariski closed, hence equals the locus defined by the equation in Appendix A.

4.2. **Exceptional Cases for $J_2 = 0$.** We consider the case when $J_2 = 0$, where

$$\begin{aligned} J_2 &= -2(3v^2 + 4u^2 - 12uv + 252u - 54v - 405) \\ &= \frac{2}{3}(3v - 6u + 2u\sqrt{6} - 27 - 18\sqrt{6})(3v - 27 + 18\sqrt{6} - 6u - 2u\sqrt{6}) \end{aligned} \tag{23}$$

Genus 2 fields K with $J_2 = J_4 = 0$ and $J_6 \neq 0$ (resp., $J_2 = J_6 = 0$ and $J_4 \neq 0$) are classified up to isomorphism by the invariant $\frac{J_6^5}{J_{10}^3}$ (resp., $\frac{J_4^5}{J_{10}^2}$), see Igusa [9]. In the first (resp., second) case there are exactly two (resp., four) such K with $e_3(K) \geq 1$ and they all have $e_3(K) = 2$.

The invariants

$$a_1 := \frac{J_4 \cdot J_6}{J_{10}}, \quad a_2 := \frac{J_6 \cdot J_{10}}{J_4^4} \tag{24}$$

determine genus two fields with $J_2 = 0$, $J_4 \neq 0$, and $J_6 \neq 0$ up to isomorphism, see Igusa [9]. A field K with such invariants has $e_3(K) \geq 1$ if and only if

$$\begin{aligned} &46656a_1^5a_2^3 + 7558272a_1^4a_2^3 + 1259712\sqrt{6}a_1^4a_2^3 - 15552a_1^4a_2^2 - 12427478784a_1^3a_2^2 + 4917635712\sqrt{6}a_1^3a_2^2 \\ &+1728a_1^3a_2 - 656217531654480a_1^2a_2^2 + 267571209034080\sqrt{6}a_1^2a_2^2 - 1844125056a_1^2a_2 + 743525568\sqrt{6}a_1^2a_2 \\ &-64a_1^2 - 6334497449472117312a_1a_2^2 + 2585860435265558832\sqrt{6}a_1a_2^2 + 230833239838992a_2a_1 - \\ &94237227087840a_2a_1\sqrt{6} - 601244429975805030777a_2^2 + 24542953925776438057 2a_2^2\sqrt{6} = 0 \end{aligned} \tag{25}$$

or

$$\begin{aligned} &230833239838992a_2a_1 + 94237227087840a_2a_1\sqrt{6} - 12427478784a_1^3a_2^2 - 4917635712\sqrt{6}a_1^3a_2^2 \\ &+1728a_1^3a_2 - 15552a_1^4a_2^2 - 656217531654480a_1^2a_2^2 - 267571209034080\sqrt{6}a_1^2a_2^2 + 46656a_1^5a_2^3 - 64a_1^2 \\ &-601244429975805030777a_2^2 - 24542953925776438057 2a_2^2\sqrt{6} - 1844125056a_1^2a_2 - 743525568\sqrt{6}a_1^2a_2 \\ &-6334497449472117312a_1a_2^2 - 2585860435265558832\sqrt{6}a_1a_2^2 + 7558272a_1^4a_2^3 - 1259712\sqrt{6}a_1^4a_2^3 = 0 \end{aligned} \tag{26}$$

As one can check by discussing the map

$$\vartheta : (u, v) \to (a_1, a_2)$$

these fields K generically have $e_3(K) = 2$. It remains to check the singular points of the curves (25) and (26). The intersection of the two components is empty. Each component has four singular points two of which have $e_3(K) = 2$ and the other two have $e_3(K) = 4$. Thus there are exactly four genus 2 fields with $e_3(K) = 4$. Two of them satisfy (25), we call them P_1 and P_2. The other two are obtained from P_1, P_2 by applying the automorphism $\sqrt{6} \to -\sqrt{6}$.

P_1	$a_1 = -\frac{77169}{8} + \frac{30759}{8}\sqrt{6}, \quad a_2 = \frac{13783592}{23149125} + \frac{5629912}{23149125}\sqrt{6}$
P_2	$a_1 = -\frac{650835}{8} + \frac{268785}{8}\sqrt{6}, \quad a_2 = -\frac{2984}{20002028625} - \frac{144872}{60006085875}\sqrt{6}$

TABLE 2. Singular points of (25) with $e_3(K) = 4$

The four values of (u, v) corresponding to P_1 (resp., P_2) are obtained as follows: u is determined by v via

$$u = \frac{1}{2}(3+\sqrt{6})(v - 9 - 6\sqrt{6})$$

and v is one of the 4 roots of

$$9v^4 + (-693 - 27\sqrt{6})v^3 + (15141 + 1749\sqrt{6})v^2 + (-66414 - 33396\sqrt{6})v \\ -1368\sqrt{6} + 3388 = 0 \tag{27}$$

(resp.)

$$3v^4 + (-153 + 33\sqrt{6})v^3 + (3129 - 1219\sqrt{6})v^2 + (-10854 - 11556\sqrt{6})v \\ +176904\sqrt{6} + 450036 = 0 \tag{28}$$

This completes the proof of the Main Theorem.

□

5. j-invariants

A genus 2 field K corresponding to a generic point of locus $e_3(K) \geq 1$ has exactly 2 elliptic subfields E_1 and E_2 of degree 3. We can take $E_1 = E$ from lemma 3. Its j-invariant is

$$j_1 = 16v\frac{(vu^2 + 216u^2 - 126vu - 972u + 12v^2 + 405v)^3}{(v-27)^3(4v^2 + 27v + 4u^3 - 18vu - vu^2)^2} \tag{29}$$

The automorphism $\beta \in Gal_{k(u,v)/k(r_1,r_2)}$ permutes E_1 and E_2, therefore switches j_1 and j_2. Then $j_2 = \beta(j_1)$ is

$$j_2 = -256\frac{(u^2 - 3v)^3}{v(4v^2 + 27v + 4u^3 - 18vu - vu^2)} \tag{30}$$

For the equation of E_2 using a different approach see [12], chapter 5.

5.1. **Isomorphic Elliptic Subfields.** Suppose that $E_1 \cong E_2$. Then, $j_1 = j_2$ implies that

$$8v^3 + 27v^2 - 54uv^2 - u^2v^2 + 108u^2v + 4u^3v - 108u^3 = 0 \tag{31}$$

or

$$324v^4u^2 - 5832v^4u + 37908v^4 - 314928v^3u - 81v^3u^4 + 255879v^3 + 30618v^3u^2 \\ - 864v^3u^3 - 6377292uv^2 + 8503056v^2 - 324u^5v^2 + 2125764u^2v^2 - 215784u^3v^2 \\ + 14580u^4v^2 + 16u^6v^2 + 78732u^3v + 8748u^5v - 864u^6v - 157464u^4v + 11664u^6 = 0 \tag{32}$$

Remark 10. *The former equation is the condition that $det(Jac(\theta)) = 0$ see equation (21).*

If $e_3(K) \geq 1$ then the automorphism group $Aut(K/k)$ is one of the following: $\mathbb{Z}_2, V_4$, D_8, or D_{12}. $\mathbb{Z}_2$ is the generic case, V_4 is a 1-dimensional family. There are exactly 6 genus 2 fields with automorphism group D_8 (resp., D_{12}). Thus, there are exactly 6 genus 2 curves (defined over $\mathbb{C}$) in the locus (33) and with automorphism group D_8 (resp., D_{12}). Exactly four (resp., three) ov these curves with automorphism group D_8 (resp., D_{12}) are defined over $\mathbb{Q}$. This is studied in [13] where the rational points of these genus 2 curves are also disscussed.

Appendix A:

Here is the equation which defines the locus $\mathcal{L}_3$ of fields K with $e_3(K) \geq 1$.

$$C_8 J_{10}^8 + C_7 J_{10}^7 + \cdots + C_1 J_{10} + C_0 = 0 \tag{33}$$

where $C_0, \ldots, C_8$ are

$C_8 = 2^4 \cdot 3^{31\cdot 5^5} \cdot 19^{10} \cdot 29^5$

$C_7 = 2^4 \cdot 3^{27} \cdot 19^5(-194894640029511J_2^5 - 55588661819356000J_4^2J_2 - 12239149540657725J_2^3J_4$
$+ 223103526505680000J_4J_6 + 40811702108053500J_2^2J_6)$

$C_6 = 2^2 \cdot 3^{21}(-358022844687577658584320J_4^5 - 17562703991065877303910J_4^2J_2^6 - 2863899185952300665 4J_4J_2^8$
$- 840912252037601594412860J_4^3J_2^4 + 400895959391006953561032J_4^4J_2^2 - 617736857389994430J_2^10$
$- 367320139607225960376516 0J_4^3J_2J_6 + 787949175521826498438720 0J_4^2J_6^2 + 152514473556086586299520J_2^5J_4J_6$
$+ 117990300838484406625027 2J_2^3J_4^2J_6 - 556667239858980988965876 0J_2^2J_4J_6^2 + 112024289372554183680J_2^7J_6$
$- 321167694097227161828880J_4^4J_6^2 + 851217187775496224915520 0J_2J_6^3)$

$C_5 = 2^2 \cdot 3^{19}(-1263000438269546265 3J_4^4J_2^7 + 320839252764287362560J_4^7J_2 - 187606927239713688644 8J_4^6J_2^3$
$+ 606742866220456356580J_4^5J_2^5 - 124173485719052715J_4^3J_2^9 + 222410345121014389440000J_4^5J_2^2J_6$
$- 885467367039388263045 12J_4^4J_2J_6^2 - 107120789264207534499 84J_2^4J_4^4J_6 + 689046353233036645112 64J_2^3J_4^3J_6^2$
$- 192353895694677164016384J_2^2J_4^2J_6^3 + 197449733923926905783808J_2J_4J_6^4 - 191617304737164522393 6J_4^6J_6$
$+ 116211018774997425051648J_4^3J_6^3 - 132143597697786172416J_6^5 + 1361403542457288J_2^10J_4J_6$
$- 50057651187404926 56J_2^7J_4J_6^2 + 232819061639483430720J_2^6J_4^3J_6 - 157631989469458517845 2J_2^5J_4^2J_6^2$
$+ 465523945920876455308 8J_2^4J_4J_6^3 - 226900590409548J_2^11J_4^2 - 2042105313685932J_2^9J_6^2 + 626163275596787280 0J_2^6J_6^3$
$- 506573479647827557617 6J_2^3J_4^6 + 1352318109350828796J_2^8J_4^2J_6)$

$C_4 = 3^{15}(1417825317153277312J_4^9J_2^2 + 2391308818408811717J_4^6J_2^8 + 718590303030600J_2^10J_4^5$
$- 6387607453371705446 40J_4J_6^6 + 440759275303802880J_4^10 - 8118717280771686540192J_4^5J_4^2J_6^2$
$+ 426684349068633980190 72J_4^2J_2J_6^5 - 570546648140206405743 36J_4^3J_2^2J_6^4 + 305467747401585816760 32J_4^4J_2^3J_6^3$
$+ 860181421512383127590 4J_4^6J_2^2J_6^2 - 144956270068219591680 0J_4^7J_2^3J_6 + 106792835412424930310 4J_4^6J_2^5J_6$
$- 104438962630243163013 12J_2^3J_4J_6^5 + 724797031515043902811 2J_2^4J_4^2J_6^4 - 217692411767517366190 08J_4^5J_2J_6^3$
$- 264020191999915459564 8J_2^5J_4^3J_6^3 - 558935624244452613 12J_2^7J_4^5J_6 + 531409635241191119304J_2^6J_4^4J_6^2$
$- 1012614205133520J_2^8J_6^4 - 245485501532619955 2J_2^5J_6^5 - 12501409939920J_2^12J_4^4 - 675076136755680J_2^10J_4^2J_6^2$
$- 33390518666828400J_2^9J_4^4J_6 + 318836356802749843 2J_2^6J_4J_6^4 - 156900149854740230 4J_2^7J_4^2J_6^3$
$- 275375222428239820800J_4^7J_6^2 + 198098490955180503306 24J_4^4J_6^4 + 617951606198374018368 0J_2^2J_6^6$
$+ 150016919279040J_2^11J_4^3J_6 + 877994814063356211 36J_4^8J_2^4 + 1350152273511360J_2^9J_4J_6^3$
$- 554966111861328006 48J_4^7J_2^6 + 399118098558425579 52J_4^8J_2J_6 + 353362680242481096J_2^8J_4^3J_6^2)$

$C_3 = 2^4 \cdot 3^{12}(-19225816442103600J_4^10J_2^5 + 6433952690394144J_2^4J_6^7 - 2917203075615J_2^11J_4^7$
$+ 62951605613640J_2^10J_4^6J_6 + 7900854051362368J_4^11J_2^3 - 8731655475519 82J_2^9J_4^8 + 132349821610444 80J_4^12J_2$
$+ 4077902864550187008J_4^5J_6^5 + 7506792545698293J_4^9J_2^7 - 55019014994202624J_4^11J_6 - 3415519987075510272J_2^4J_6^7$
$- 932605137272623104J_4^8J_6^3 - 6607177263254292480J_2J_6^8 - 1394785406520J_2^7J_6^6 - 1913285880J_2^13J_4^6$
$- 258293593800J_2^11J_4^4J_6^2 - 19762959576163015 04J_4^8J_2^3J_6^2 + 13375981920580417 44J_4^7J_2^5J_6^2$
$- 2324642344200J_2^9J_4^2J_6^4 + 2789570813040J_2^8J_4J_6^5 - 243015467955111198J_2^7J_4^6J_6^2 + 221367618013486 68J_2^8J_4^7J_6$
$- 155463896437263612J_4^8J_2^6J_6 + 161010337961830043 52J_4^5J_2^3J_6^4 + 163672986317964503 04J_4^3J_2J_6^6 + 8433152J_4^7J_2^2J_6^3$
$- 825496517802146918 4J_4^6J_2J_6^4 + 34439145840J_2^12J_4^5J_6 - 5769881306823 78J_2^9J_4^5J_6^2 + 2912934238489260J_2^8J_4^4J_6^3$
$- 87498754124541 75J_2^7J_4^3J_6^4 + 156375115922003 40J_2^6J_4^2J_6^5 - 127105068829245696J_4^10J_2^2J_6 + 614908581517421568J_4^9J_2J_6^2$
$- 233744194313602076 16J_4^4J_2^2J_6^5 + 150886894860594698 4J_2^6J_4^5J_6^3 - 57950402944706238 24J_2^5J_4^4J_6^4$
$+ 140949838965116301 12J_2^4J_4^3J_6^5 + 1033174375200J_2^10J_4^3J_6^3 + 3140697982040694 72J_4^9J_2^4J_6 - 61501104J_4^6J_2^4J_6^3$
$- 211941630802220250 24J_2^3J_4^2J_6^6 + 180024021191763325 44J_2^2J_4J_6^7 - 15392091937240080J_2^5J_4J_6^6)$

$C_2 = 2^5 \cdot 3^8(-159732958548480J_4^13J_2J_6 - 279451929685939 20J_2J_4J_6^9 + 238596124150086J_4^10J_2^7J_6 + 3224288J_2^3J_6^9$
$- 363111362152 44J_2^9J_4^9J_6 - 996173640J_2^12J_4^6J_6^2 + 5977041840J_2^11J_4^5J_6^3 - 22413906900J_2^10J_4^4J_6^4 - 37501414009508J_4^13J_2^4$

$$+14210312049697149J_2^6J_4^6J_6^4+86354918885580768J_2^4J_4^4J_6^6-111444977082978432J_2^3J_4^3J_6^7+27908893977856J_4^{14}J_2^2$$
$$+83768141083825152J_2^2J_4^2J_6^8-42942980968765488J_2^5J_4^5J_6^5-10736445647473J_4^{11}J_2^8+61746352553318400J_4^4J_2J_6^7$$
$$+410958880454688J_4^{12}J_2^3J_6-140592520576604 16J_4^3J_6^8-1643659809866496J_4^{11}J_2^2J_6^2+441832778741790J_2^8J_4^8J_6^2$$
$$-31285995511086 36J_2^7J_4^7J_6^3+28159504954306 56J_4^{10}J_2J_6^3+12291244885171152J_4^8J_2^5J_6^3+1579225145J_2^{12}J_4^9$$
$$-2286353789913249J_4^9J_2^6J_6^2-40300476525629352J_4^7J_2^4J_6^4+81707043798929088J_4^6J_2^3J_6^5-25936092270J_2^6J_6^8$$
$$-98257765274489088J_4^5J_2^2J_6^6+33188872074808 32J_4^{10}J_2^4J_6^2-478511899451856J_4^{11}J_2^5J_6+284762870516 77J_4^12J_2^6$$
$$-12153253649302656J_4^9J_2^3J_6^3+24757975700165376J_4^8J_2^2J_6^4-26570902457981952J_4^7J_2J_6^5-6755065089024J_4^{15}$$
$$+10883911680J_6^{10}+53793376560J_2^9J_4^3J_6^5-80690064840J_2^8J_4^2J_6^6+69162912720J_2^7J_4J_6^7+94873680J_2^{13}J_4^7J_6$$
$$+227109129291J_2^{10}J_4^7J_6^2-628213747356J_2^9J_4^6J_6^3-1389130574661J_2^8J_4^5J_6^4+164657939888 70J_2^7J_4^4J_6^5$$
$$-56794191944715J_2^6J_4^3J_6^6+102713329135152J_2^5J_4^2J_6^7-98529746457492J_2^4J_4J_6^8-30650938650J_2^{11}J_4^8J_6$$
$$-1716480768J_4^9J_6^4+11718053954519040J_4^6J_6^6+220752428322816J_4^{12}J_6^2+1322792799725J_2^{10}J_4^{10}-3953070J_2^{14}J_4^8)$$

$$C_1=-2^8\cdot3^5(61736960J_4^8J_2-182135808J_4^7J_6+16021872J_4^7J_2^3-211022400J_4^6J_2^2J_6-26594919J_4^6J_2^5$$
$$+899159040J_4^5J_2J_6^2+330458928J_4^5J_4^2J_6-215198J_2^7J_4^5-1227405312J_4^4J_6^3-1535734368J_4^4J_2^3J_6^2$$
$$+2930532J_4^4J_2^6J_6-363J_2^9J_4^4+3162070656J_4^3J_2^2J_6^3-16471998J_4^3J_2^5J_6^2+4356J_4^3J_2^8J_6-19602J_4^2J_2^7J_6^2$$
$$-2433162240J_4^2J_2J_6^4+47961936J_4^2J_2^4J_6^3+39204J_4J_2^6J_6^3-72369936J_4J_2^3J_6^4+746496J_4J_6^5-29403J_2^5J_6^4$$
$$+45116352J_2^2J_6^5)(J_4^3-J_2^2J_4^2+6J_2J_6J_4-9J_6^2)^3$$

$$C_0=2^8(768J_4^2-416J_4J_2^2-J_2^4+1536J_2J_6)(J_4^3-J_2^2J_4^2+6J_2J_6J_4-9J_6^2)^6$$

Appendix B:

The equation of the branch locus of the map

$$\theta: k^2\setminus\{\Delta=0\}\to\mathcal{L}_3$$
$$(u,v)\to(i_1,i_2,i_3)$$

(34)

$$3507505273398025i_1^{12}-48804848177938620735484 80i_1^{11}+192(113025049383884896281 25i_2$$
$$+34645203923768965058119 2)i_1^{10}-4(89439905046278964319358976+6030715704603053299722 5i_2^2$$
$$-41889810660691132346486 40i_2)i_1^9+192(49246723558095680048855 04+33588887753890413143515i_2^2$$
$$-89290373874540245356608i_2)i_1^8-192(63601395912353833813278 72+59358780513584507843863 2i_2^2$$
$$+84939767888511469682995 2i_2-15537701670163340329775i_2^3)i_1^7+2(40660607474296284191628902 4i_2^2$$
$$-27711519511099200420652160i_2^3+30552600634759635629034700 8+18304980860695577479407 5i_2^4$$
$$+25532076531322078257689395 2i_2)i_1^6-192i_2(13654023711946692280289280i_2+60377228453350507376315i_2^3$$
$$-14643500227714429972657 92i_2^2+22457545304665379297034 24)i_1^5+192i_2^2(19906166147692916436664320$$
$$+52446584699177845511743 2i_2^2-23333601385906920314818 56i_2-10694743521252049023175i_2^3)i_1^4$$
$$+20i_2^2(25359283200486542884819 84i_2^3-57636649269631838453760000i_2-28262691566657845506883584i_2^2$$
$$-99413657833844087193600000-92803044099032574023 25i_2^4)i_1^3-192i_2^3(18076023264213617319826 56i_2^2$$
$$-76401752022567738591625i_2^3-28676270196566138880000 00-99189527154240669959116 80i_2)i_1^2$$
$$+1728i_2^4(12623712307082454341 25i_2^3-44970318919363276752280i_2^2-17528823669935877120000 00$$
$$+50687044955460223588454 4i_2)i_1+i_2^4(12738539564043290937562 5i_2^4-60597651352868379683776 00i_2^3$$
$$-71714832485837325926400000 0i_2+10283838754777259864135270 4i_2^2+1617386925920256\cdot10^{12})=0$$

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