THE LOCUS OF CURVES WITH PRESCRIBED AUTOMORPHISM GROUP

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Abstract. Let G be a finite group, and $g \geq 2$. We study the locus of genus g curves that admit a G-action of given type, and inclusions between such loci. We use this to study the locus of genus g curves with prescribed automorphism group G. We completely classify these loci for $g = 3$ (including equations for the corresponding curves), and for $g \leq 10$ we classify those loci corresponding to "large" G.

1. Introduction

There is a vast literature on automorphism groups of compact Riemann surfaces, beginning in the 19th century with Schwartz, Klein, Hurwitz, Wiman and others. However, most of the literature is quite recent. In the first part of the paper, we survey the main results.

By covering space theory, a finite group G acts (faithfully) on a genus g curve if and only if it has a genus g generating system (see section section5 below). Using this purely group-theoretic condition, Breuer [?Br] classified all groups that act on a curve of genus ≤ 48. This was a major computational effort using the computer algebra system GAP [?GAP]. It greatly improved on several papers dealing with small genus, by various authors (see the references in part I).

Of course, for each group in Breuer's list, all subgroups are also in the list. This raises the question how to pick out those groups that occur as the **full automorphism group** of a genus g curve. This question is answered in Part II of the paper. Let $\mathcal{M}_g$ be the moduli space of genus g curves. We study the locus $\mathcal{L}$ in $\mathcal{M}_g$ of curves admitting a G-action of given ramification type (resp., signature). All components of $\mathcal{L}$ have the same dimension which depends only on the signature of the G-action. Restricting the action to a subgroup H of G yields an inclusion of $\mathcal{L}$ into the corresponding locus $\mathcal{L}'$ for the action of H. If $\dim \mathcal{L} < \dim L'$ then for "most" curves in $\mathcal{L}'$, the H-action does not extend to a G-action of the type defining $\mathcal{L}$. Thus one is led to classify the pairs $(\mathcal{L}, \mathcal{L}')$ with $\dim \mathcal{L} = \dim \mathcal{L}'$. This is done in Lemma Lemma4.1 below: It turns out that such pairs exist only in very restricted cases, in particular only if $\dim \mathcal{L} \leq 3$. From that we derive a necessary and sufficient condition for a group to occur as the full automorphism group of a genus g-curve (Theorem **??**).

After Part II was written, we found references to papers of Singerman [?Si] and Ries [?Ri] which contain similar results. Their method is different, of analytic nature, using Teichmüller theory and Fuchsian groups. Our approach is based on algebraic geometry, using the algebraic structure of Hurwitz spaces and the moduli spaces $\mathcal{M}_g$; therefore, it can be used to obtain information on fields of definition. This aspect may be studied in later work.

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In Part III we apply the above criterion to the data compiled by Breuer (available from his website, see 1.5 below). Our first application is in the case of genus 3, which is already quite rich and shows the power of our group-theoretic method. We obtain the full list of automorphism groups in genus 3 plus equations for the corresponding curves. This result is scattered over several long papers by Kuribayashi and various co-authors [?KuKo1], [?KuKo2], [?KK1], [?KK2], some of which contain errors (see section 6 below) and some are not available in most libraries. None of them seems to give a complete account of this basic result. Here we show how to derive it quickly from the group-theoretic data.

Further, we obtain the list of "large" groups $\mathrm{Aut}(X_g)$ (see 1.3 below) up to $g = 10$ and the dimension and number of components of the corresponding loci in $\mathcal{M}_g$, see Table 4.

The locus of curves in $\mathcal{M}_g$ with fixed automorphism group consists of finitely many components; to determine their number requires tools that will be developed in later work (mapping class group action on generating systems). In the special case of components corresponding to curves of orbit genus 0 (i.e., the quotient by the automorphism group of the curve has genus 0), the mapping class group action is just the braid group action studied in [?FV], [?V], [?Buch]; and those components correspond to braid orbits on generating systems, which can be computed with our GAP package BRAID [?MSV]. A curve with large automorphism group always has orbit genus 0, so we were able to use the BRAID package to compute the information in Table 4. The details of this computation will be given in later work (in a more general situation).

Notation: We will use the term "curve" to mean "compact Riemann surface"; and $X = X_g$ denotes a curve of genus $g \geq 2$. Further, $\mathrm{Aut}(X)$ denotes the group of analytic (equivalently, algebraic) automorphisms of X.

Part 1. Survey of known results

1.1. **The beginnings in the 19th century.** The group $\mathrm{Aut}(X)$ acts on the finite set of Weierstraß points of X. This action is faithful unless X is hyperelliptic, in which case its kernel is the group of order 2 containing the hyperelliptic involution of X. Thus in any case, $\mathrm{Aut}(X)$ is a finite group. This was first proved by Schwartz.

F. Klein [?Kl1] (1879) studied the genus 3 projective curve

$$x^3y + y^3z + z^3x = 0 \tag{1}$$

which is now known as Klein's quartic. He found all its automorphisms: They form the simple group of order 168. Further, he showed that there is a degree 168 covering from this curve to $\mathbb{P}^1$ branched over three points with ramification indices 2,3,7. See [?Le] for a detailed history on Klein's quartic (including an English translation of Klein's original article) as well as an account of further developments influenced by it.

Poincaré introduced Fuchsian groups in 1882. They are basic for the analytic theory of Riemann surfaces and have been used heavily in the study of $\mathrm{Aut}(X)$ (see [?Br], [?Ri] for example). In this paper we take a different (algebraic) point of view.

The next milestone was Hurwitz's seminal paper [?Hu] in 1893, where he discovered what is now called the Riemann-Hurwitz formula. From this he derived that $|\mathrm{Aut}(X_g)| \leq 84\,(g-1)$, now known as the Hurwitz bound. This bound is attained by Klein's quartic, for example. Curves which attain this bound are called **Hurwitz curves** and their automorphism groups **Hurwitz groups**. Hurwitz proved that a finite group is a Hurwitz group if and only if it has generators a, b, c of orders $2, 3, 7$, respectively, with $abc = 1$.

1.2. **Hurwitz groups.** Klein's quartic is the only Hurwitz curve of genus $g \leq 3$. Fricke showed that the next Hurwitz group occurs for $g = 7$ and has order 504. Its group is $SL(2, 8)$, and an equation for it was computed by Macbeath [?Mb2] in 1965. Klein's quartic and Macbeath's curve

are the only Hurwitz curves whose equations are known. Further Hurwitz curves occur for $g = 14$ and $g = 17$ (and for no other values of $g \leq 19$).

There are a lot of papers by group-theoretists on Hurwitz groups, surveyed by Conder [?Co]. It follows from Hurwitz's presentation that a Hurwitz group is **perfect**. Thus every quotient is again a Hurwitz group, and if such a quotient is minimal then it is a non-abelian simple group. Several infinite series of simple Hurwitz groups have been found by Conder, Malle, Kuribayashi, Zalessky, Zimmermann and others. In 2001, Wilson [?Wl] showed the monster is a Hurwitz group.

1.3. **Large automorphism groups.** For a fixed $g \geq 2$ denote by $N(g)$ the maximum of the $|\mathrm{Aut}(X_g)|$. Accola [?Ac1] and Maclachlan [?Mc1] independently show that $N(g) \geq 8(g+1)$ and this bound is sharp for infinitely many g's. If g is divisible by 3 then $N(g) \geq 8(g+3)$.

The following terminology is rather standard. We say $G \leq \mathrm{Aut}(X_g)$ is a **large automorphism group** in genus g if $|G| \quad > \quad 4(g-1)$. Then the quotient of X_g by G is a curve of genus 0, and the number of points of this quotient ramified in X_g is 3 or 4 (see [?Br], Lemma 3.18, or [?FK], pages 258-260). Singerman [?S1] (1974) shows that Riemann surfaces with large cyclic, Abelian, or Hurwitz groups are symmetric (admit an involution). Kulkarni [?Ku](1997) classifies Riemann surfaces admitting large cyclic automorphism groups and works out full automorphism groups of these surfaces. Matsuno [?Mt](1999) investigates the Galois covering of the projective line from compact Riemann surfaces with large automorphism groups.

1.4. **Cyclic groups as automorphism groups.** Let t be the order of an automorphism of X_g. Hurwitz [?Hu] showed $t \leq 10(g-1)$. In 1895, Wiman improved this bound to be $t \leq 2(2g+1)$ and showed this is best possible. If t is a prime then $t \leq 2g+1$. Homma [?Ho] (1980) shows that this bound is achieved if and only if the curve is birationally equivalent to

$$y^{m-s}(y-1)^s = x^q, \quad for \quad 1 \leq s < m \leq g_x + 1$$

1.5. **The canonical representation of Aut(X).** Each group $G \leq \mathrm{Aut}(X_g)$ acts faithfully on the g-dimensional vector space of holomorphic differential forms on X. The resulting finite subgroups of $GL(g, \mathbb{C})$ satisfy a number of special conditions, studied by I. Kuribayashi, A. Kuribayashi, Kimura, Ohmori, Kobayashi, Hayakawa and others. A. Kuribayashi and others (see [?KuKi1] and the references in that paper) have completely listed these finite linear groups up to genus 5.

Using computational group theory (more precisely, the computer algebra system GAP [?GAP]), Breuer [?Br] extended this up to genus 48: Computing all groups $G \leq \mathrm{Aut}(X_g)$ together with their character on the space of holomorphic differential forms. He avoids ad-hoc methods by using the GAP-library of "small groups" by Besche and Eick [?BE]. His results are collected in a database which is available from his website at www.math.rwth-aachen.de/ Thomas.Breuer/. The GAP library of small groups is available at http://www-public.tu-bs.de:8080/~beick/so.html.

In our tables a group G is identified by its Group ID from the Small Groups Library. A Group ID is a pair (n, m) where n is the order of G and m the number of G among the groups of order n.

1.6. **Full automorphism groups.** For each group in Breuer's list, all subgroups are also in the list. This raises the question how to pick out those groups that occur as the **full automorphism group** of a genus g curve. Answering this question requires a moduli space argument: Either analytically, using Fuchsian groups and Teichmüller space, as done by Singerman [?Si] and Ries [?Ri], or algebraically, as done in Part II of this paper. Using these general results, one can compute explicitly all groups $\mathrm{Aut}(X_g)$ for small g. We give the complete list for $g = 3$ in Part III, and the list of large groups $\mathrm{Aut}(X_g)$ up to $g = 10$. Further results will be given in later work, together with additional information.

For $g > 3$, there are no lists of full automorphism groups in the literature. The case $g = 2$ is easy, but already for $g = 3$ there is no complete account (see the remarks in the introduction).

Part 2. Subgroups of $\mathbf{Aut}(X_g)$ not occurring as full automorphism groups in genus g

2. Ramification type and signature of a G-curve

Fix an integer $g \geq 2$ and a finite group G. Let $C_1, ..., C_r$ be conjugacy classes $\neq \{1\}$ of G. Let $\mathbf{C} = (C_1, ..., C_r)$, viewed as unordered tuple, repetitions are allowed. We allow r to be zero, in which case $\mathbf{C}$ is empty.

We will use the term "curve" to mean "compact Riemann surface". Consider pairs (X, μ), where X is a curve and $\mu : G \to \mathrm{Aut}(X)$ is an injective homomorphism. Mostly we will suppress μ and just say X is a curve with G-action, or a G-curve, for short. Two G-curves X and X' are called equivalent if there is a G-equivariant isomorphism $X \to X'$.

We say a G-curve X is **of ramification type** $(g, G, \mathbf{C})$ if the following holds: Firstly, g is the genus of X. Secondly, the points of the quotient X/G that are ramified in the cover $X \to X/G$ can be labelled as $p_1, ..., p_r$ such that C_i is the conjugacy class in G of distinguished inertia group generators over p_i (for $i = 1, ..., r$). (Distinguished inertia group generator means the generator that acts in the tangent space as multiplication by $\exp(2\pi\sqrt{-1}/e)$, where e is the ramification index). For short, we will just say X is of type $(g, G, \mathbf{C})$.

If X is a G-curve of type $(g, G, \mathbf{C})$ then the genus g_0 of X/G is given by the Riemann-Hurwitz formula

$$\text{(1)}, \qquad \frac{2\,(g-1)}{|G|} \quad = \quad 2\,(g_0 - 1) \;+\; \sum_{i=1}^{r} \left(1 - \frac{1}{c_i}\right)$$

where c_i is the order of the elements in C_i.

Note that g_0 (the **orbit genus**) depends only on g, $|G|$ and the **signature** $\mathbf{c} = (c_1, \ldots, c_r)$ of the G-curve X.

3. Hurwitz spaces and moduli of curves

Define $\mathcal{H} = \mathcal{H}(g, G, \mathbf{C})$ to be the set of equivalence classes of G-curves of type $(g, G, \mathbf{C})$. By covering space theory (or the theory of Fuchsian groups), $\mathcal{H}$ is non-empty if and only if G can be generated by elements $\alpha_1, \beta_1, ..., \alpha_{g_0}, \beta_{g_0}, \gamma_1, ..., \gamma_r$ with $\gamma_i \in C_i$ and

$$\text{(2)} \qquad \prod_j [\alpha_j, \beta_j] \;\; \prod_i \gamma_i \quad = \quad 1$$

Here $[\alpha, \beta] = \; \alpha^{-1}\beta^{-1}\alpha\beta$.

Let $\mathcal{M}_g$ be the moduli space of genus g curves, and $\mathcal{M}_{g_0,r}$ the moduli space of genus g_0 curves with r distinct marked points, where we view the marked points as unordered (contrary to usual procedure). Consider the map

$$\Phi : \; \mathcal{H} \; \to \; \mathcal{M}_g$$

forgetting the G-action, and the map

$$\Psi : \; \mathcal{H} \; \to \; \mathcal{M}_{g_0,r}$$

mapping (the class of) a G-curve X to the class of the quotient curve X/G together with the (unordered) set of branch points $p_1, ..., p_r$. If $\mathcal{H} \neq \emptyset$ then Ψ is surjective and has finite fibers, by covering space theory. Also Φ has finite fibers, since the automorphism group of a curve of genus ≥ 2 is finite.

By [?B], the set $\mathcal{H}$ carries a structure of quasi-projective variety (over $\mathbb{C}$) such that the maps Φ and Ψ are finite morphisms. If $\mathcal{H} \neq \emptyset$ then all components of $\mathcal{H}$ map surjectively to $\mathcal{M}_{g_0,r}$ (through a finite map), hence they all have the same dimension

$$\delta(g, G, \mathbf{C}) := \quad \dim \; \mathcal{M}_{g_0,r} \quad = \quad 3g_0 - 3 + r$$

(This is correct because $r \geq 1$ if $g_0 = 1$ and $r \geq 3$ if $g_0 = 0$; the latter holds because $g \geq 2$). Since also Φ is a finite map, we get

Lemma 3.1. *Let $\mathcal{M}(g, G, \mathbf{C})$ denote the image of Φ, i.e., the locus of genus g curves admitting a G-action of type $(g, G, \mathbf{C})$. If this locus is non-empty then each of its components has dimension $\delta(g, G, \mathbf{C})$.*

Note that $\delta(g, G, \mathbf{C})$ depends only on g, $|G|$ and the signature, so we write $\delta(g, G, \mathbf{c}) := \delta(g, G, \mathbf{C})$. The Lemma continues to hold if we replace $\mathbf{C}$ by $\mathbf{c}$.

4. Restriction to a subgroup

Let H be a subgroup of G. Then each G-curve can be viewed as an H-curve by restriction of action. Let X be a G-curve of type $(g, G, \mathbf{C})$. Then the resulting H-curve is of type $(g, H, \mathbf{D})$, where $\mathbf{D}$ is obtained as follows: Choose $\gamma_i \in C_i$ and let $\sigma_{i,1}, \sigma_{i,2}, ...$ be a set of representatives for the double cosets $< \gamma_i > \sigma H$ in G. Let m_{ij} be the smallest integer ≥ 1 such that the element $\sigma_{ij}^{-1}\gamma_i^{m_{ij}}\sigma_{ij}$ lies in H, and let D_{ij} be the conjugacy class of this element in H. Then $\mathbf{D}$ is the tuple consisting of all D_{ij} with $D_{ij} \neq \{1\}$. (More precisely, the tuple $\mathbf{D}$ is indexed by the set of possible pairs (i, j), and its (i, j)-entry is D_{ij}). The definition of $\mathbf{D}$ does not depend on the choice of the γ_i and σ_{ij}. Note that the signature of the H-curve depends on the type of the G-curve, not only on its signature.

We have

$$\mathcal{M}(g, G, \mathbf{C}) \quad \subset \quad \mathcal{M}(g, H, \mathbf{D})$$

hence their dimensions satisfy

$$\delta(g, G, \mathbf{C}) \quad \leq \quad \delta(g, H, \mathbf{D})$$

If this is a strict inequality then the complement of the closure of $\mathcal{M}(g, G, \mathbf{C})$ in $\mathcal{M}(g, H, \mathbf{D})$ is open and dense; then in particular, it is not true that every H-curve of type $(g, H, \mathbf{D})$ is the restriction of a G-curve of type $(g, G, \mathbf{C})$.

Lemma 4.1. *Suppose H is a subgroup of G of index $n > 1$. We let $\bar{G}$ denote the permutation group induced by G on the coset space G/H. Let X be a G-curve of type $(g, G, \mathbf{C})$, where we label the C_i such that $c_1 \leq \ldots \leq c_r$. Let $\mathbf{c} = (c_1, \ldots, c_r)$. The H-curve obtained by restriction of action is of type $(g, H, \mathbf{D})$, where $\mathbf{D}$ is defined above; let $(g, H, \mathbf{d})$ be its signature. Let s (resp., r) be the length of $\mathbf{D}$ (resp., $\mathbf{C}$). Let h_0 (resp., g_0) be the genus of X/H (resp., X/G). If $\delta := \delta(g, G, \mathbf{C}) = \delta(g, H, \mathbf{D})$ then $g_0 = 0$, and one of the following holds:*

(I) $\delta = 3$: *Then $n = 2$, $h_0 = 2$, $s = 0$, $r = 6$, all $c_i = 2$ and $C_i \not\subset H$.*

(II) $\delta = 2$: *Then $n = 2$, $h_0 = 1$, $s = 2$, $r = 5$, $C_i \not\subset H$ and $c_i = 2$ for $i = 1, ..., 4$ and $\mathbf{d} = (c_5, c_5)$.*

(III) $\delta = 1$:

- (a) *$n = 2$, $h_0 = 1$, $s = 1$, $r = 4$, $C_i \not\subset H$ for all i, $c_1 = c_2 = c_3 = 2$ and $\mathbf{d} = (c_4/2)$.*
- (b) *$n = 2$, $h_0 = 0$, $s = 4 = r$, $c_1 = c_2 = 2$, $C_3, C_4 \subset H$, $\mathbf{d} = (c_3, c_3, c_4, c_4)$.*
- (c) *$h_0 = 0$, $s = 4 = r$, $c_1 = c_2 = c_3 = 2$, $C_4 \subset H$, and H is normal in G with G/H a Klein 4-group; and $\mathbf{d} = (c_4, c_4, c_4, c_4)$.*

(IV) $\delta = 0$: *Then $h_0 = 0$, $s = 3 = r$ and*

- (a) *$n = 2$, $h_0 = 0$, $r = 3 = s$, $c_1 = 2$, $c_i > 2$ is even and $C_j \subset H$ where $\{i, j\} = \{2, 3\}$; and $\mathbf{d} = (c_i/2, c_j, c_j)$.*
- (b) *$n = 3$, $\bar{G} \cong \mathbb{Z}/3$, one $C_i \subset H$ and the other two have $c_j = 3$; here $\mathbf{d} = (c_i, c_i, c_i)$.*
- (c) *$n = 3$, $\bar{G} \cong S_3$, $c_1 = 2$, $c_2 = 3$, $c_3 > 2$ is even, $\mathbf{d} = (2, c_3/2, c_3)$.*
- (d) *$n = 4$, $\bar{G} \cong A_4$, $c_1 = 2$, $c_2 = 3$, $c_3 > 3$ is divisible by 3, $\mathbf{d} = (3, c_3/3, c_3)$.*
- (e) *$n = 6$, $\bar{G} \cong PGL_2(5)$, $\mathbf{c} = (2, 4, 5)$, $\mathbf{d} = (4, 4, 5)$, $g \equiv 1$ mod 3.*
- (f) *$n = 8$, $\bar{G} \cong PSL_2(7)$, $\mathbf{c} = (2, 3, 7)$, $\mathbf{d} = (3, 3, 7)$, $g \equiv 1$ mod 2.*
- (g) *$n = 9$, $\bar{G} \cong PSL_2(8)$, $\mathbf{c} = (2, 3, 7)$, $\mathbf{d} = (2, 7, 7)$, $g \equiv 1$ mod 6.*
- (h) *$n = 10$, $\bar{G} \cong PGL_2(9)$, $\mathbf{c} = (2, 3, 8)$, $\mathbf{d} = (3, 8, 8)$, $g \equiv 1$ mod 15. $g \equiv 1$*
- (i) *There are 6 further cases when H is not maximal in G, and $n = 4, 6, 6, 12, 12, 24$, respectively.*

The cases in (IV) are taken from [?M], Table 2 (c.f. 5b below).

Proof : We denote the hypothesis $\delta(g,G,\mathbf{C}) = \delta(g,H,\mathbf{D})$ by (Hyp). Let r_1 be the number of points of X/G that are ramified in X/H. Applying Riemann-Hurwitz to the cover $X/H \to X/G$ of degree n yields

$$2\ (h_0-1) \ \geq \ 2n\ (g_0-1)\ +\ r_1 \tag{3}$$

Let s be the number of points of X/H that are ramified in X, and $r_2 := r - r_1$. The r_2 points of X/G that are ramified in X, but not in X/H, have a total of nr_2 distinct pre-images in X/H. The latter points are ramified in X, hence $s \geq nr_2$.

Case 1: $g_0 > 1$: Then (Hyp) gives

$$3g_0\ +\ r_1\ +\ r_2 \ = \ 3h_0\ +\ s \ \geq \ 3h_0\ +\ r_2$$

hence $3g_0\ +\ r_1 \ \geq \ 3h_0$. With (3) this gives

$$3g_0\ +\ 2\ (h_0-1)\ -\ 2n\ (g_0-1) \ \geq \ 3h_0$$
$$g_0\ (3-2n)\ +\ 2n \ \geq \ h_0\ +\ 2 \ \geq \ g_0\ +\ 2$$
$$g_0 \ < \ \frac{n}{n-1} \ \leq \ 2$$

a contradiction.

Case 2: $g_0 = 1$: If $r = 0$ then by (1) we have $g = 1$, contrary to assumption. Thus $r > 0$. Now (3) gives $2\ (h_0-1) \geq r_1$. If $h_0 > 1$ then (Hyp) gives $r_1 \geq\ 3(h_0-1)$ (as in Case 1), a contradiction. If $h_0 = 1$ then $r_1 = 0$, hence (Hyp) gives $r_2 = s \geq nr_2$. Contradiction, since $r_2 = r > 0$.

We have now proved $g_0 = 0$. Thus if $h_0 > 0$ then $r_1 \geq 3$. Further, we can choose $\gamma_i \in C_i$ with $\gamma_1 \cdots \gamma_r = 1$ and $G =< \gamma_1, ..., \gamma_r >$ (by (2)). Then c_i is the order of γ_i. Let $p_1, ..., p_r$ (resp., $q_1, ..., q_s$) be the points of X/G (resp., X/H) that are ramified in X. We choose the labelling such that p_i corresponds to C_i and

$$c_1 \ \leq \ \ldots \ \leq \ c_r \tag{4}$$

The orbits of γ_i on G/H correspond to the points of X/H over p_i (where the length of the orbit equals the ramification index of the corresponding point over p_i). Let $\mathrm{Ind}(C_i)$ (=$\mathrm{Ind}(\gamma_i)$) be n minus the number of those orbits. Those orbits that correspond to points $\notin \{q_1, ..., q_s\}$ have length $c_i \geq 2$. Thus the total number of orbits of all γ_i is $\leq s + (nr - s)/2$. Hence Riemann-Hurwitz for $X/H \to X/G$ yields

$$2\ (h_0 - 1 + n) \ = \ \sum_{i=1}^{r} \mathrm{Ind}(C_i) \ \geq \ nr\ -\ \frac{nr+s}{2} \ = \ \frac{nr-s}{2} \tag{5}$$

Case 3: $h_0 \geq 2$: Here (Hyp) gives $r = 3h_0 + s$. Plugging this into (5) gives

$$2\ (h_0 - 1 + n) \ \geq \ \frac{n\ (3h_0+s)\ -\ s}{2} \ = \ \frac{3}{2}h_0 n\ +\ \frac{s(n-1)}{2} \ \geq \ \frac{3}{2}h_0 n \tag{6}$$

This yields $n \leq 2(h_0-1)/(\frac{3}{2}h_0 - 2) \leq 2$ since $h_0 \geq 2$. Plugging $n = 2$ into (6) gives $h_0 \leq 2$. Thus $h_0 = 2$ and the inequalities in (6) are equalities, which implies $s = 0$. This is case (I) in the Lemma.

Case 4: $h_0 = 1$: If $s = 0$ then $g = 1$, contrary to assumption. Thus $s > 0$, and so (Hyp) gives $r - 3 = s$. Thus $r \geq 4$. Plugging $s = r - 3$ into (5) gives

$$2n \ \geq \ \frac{r(n-1)\ +\ 3}{2} \tag{7}$$

which implies $r \leq\ 4 + \frac{1}{n-1}$. If $n = 2$ then we are in case (II) resp. (III).a of the Lemma. Assume now $n > 2$. Then $r = 4$, $s = 1$. If all $c_i = 2$ then (7) would have to hold with equality, which is not possible. Thus $c_4 \geq 3$ (by (4)).

Let p_{i_0} be the image in X/G of q_1. If $i_0 < 4$ then Riemann-Hurwitz yields $2n \geq \frac{3n-1}{2} + \frac{2}{3}n$, hence $n = 3$ and $c_1 = c_2 = c_3 = 2$, a contradiction (since n is divisible by those c_i with $i \neq i_0$). Thus $i_0 = 4$. If $n = 3$ then $c_1 = c_2 = c_3 = 3$ contradicting Riemann-Hurwitz. Thus $n > 3$. Riemann-Hurwitz yields $2n \geq 3\,\frac{n}{2} + \frac{2}{3}(n-1)$, hence $n = 4$, $c_1 = c_2 = c_3 = 2$ and $c_4 = 3$. Thus γ_1, γ_2,γ_3 act as double transpositions on the 4-set G/H, and γ_4 acts as 3-cycle. However, three double transpositions in S_4 cannot have a 3-cycle as their product (since the double transpositions together with the identity form a normal subgroup). Contradiction.

Case 5: $h_0 = 0$: If $r \leq 2$ or $s \leq 2$ then $g = 0$, contradiction. Thus we have $r \geq 3$ and $s \geq 3$. Hence (Hyp) gives $r = s$. Plugging this into (5) gives $r \leq 4$.

Case 5.a: $r = 4 = s$: Here (5) holds with equality, hence the q_j's are unramified over X/G, and all other points of X/H over some p_i have ramification index $2 = c_i$. If all $c_i = 2$ then (1) gives $g = 1$, contradiction. Thus some p_i, say p_{i_0}, is unramified in X/H. If two p_i's are unramified in X/H then $s \geq 2n$, hence $n \leq s/2 = 2$; this is case III.b in the Lemma.

Now suppose p_i is unramified in X/H only for $i = i_0$. Then $c_i = 2$ for $i \neq i_0$, and n is even (since not each p_i, $i \neq i_0$, can have a q_j over it); also, $n \leq s = 4$ and $n \neq 2$ (because $X/H \to X/G$ has 3 branch points). Hence $n = 4$ and the γ_i with $i \neq i_0$ act as double transpositions on the 4-set G/H (and γ_{i_0} acts trivially). This is case III.c.

Case 5.b: $r = 3 = s$: This is the last case to be considered. Now X/H and X/G have genus zero, and the cover $X/H \to X/G$ (resp., $X \to X/H$) is ramified at 3 points of X/G (resp., of X/H). This gives the following condition on the action of $\gamma_1, \gamma_2, \gamma_3$ on the n-set G/H: Apart from exactly three orbits of $\gamma_1, \gamma_2, \gamma_3$, all orbit lengths of γ_i equal c_i; and if one of the three exceptional orbits belongs to γ_i then its length is $< c_i$. From (1) we also get $\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} < 1$. Finally, we have the genus zero condition $2(n-1) = \sum_{i=1}^{3} \mathrm{Ind}(\gamma_i)$. Triples of permutations with these properties (and with product 1, and generating a transitive group) have been classified by Malle [?M]: His Table 2 gives the cycle types of the three permutations plus the group they generate (plus the c_i). This is part (IV) of the Lemma. (Actually, Malle's situation is slightly more general than ours, since he doesn't require c_i to be the order of γ_i; thus two of his cases, where $n = 10$ and $n = 18$, don't occur here). ∎

5. The full automorphism group

Now we change perspective and fix the finite group H. An H-curve X is called **generic** if the following holds: The curve X/H, together with its points $q_1, ..., q_s$ that are ramified in X, defines a generic point of $\mathcal{M}_{h,s}$. Here h is the genus of X/H, and 'generic' means generic over $\overline{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$).

Let $\mathrm{Aut}(X)$ denote the automorphism group of the curve X (without regard of H-action).

Theorem 5.1. *Suppose X is a generic H-curve of genus ≥ 2. Then $[Aut(X) : H] \leq 24$. Let h be the genus of X/H, and s the number of points of X/H ramified in X. Then $Aut(X) = H$ unless (h, s) equals $(2, 0)$, $(1, 2)$, $(1, 1)$, $(0, 4)$ or $(0, 3)$.*

Then there is an integer $n \leq 24$ such that $[Aut(Y) : H] \leq n$ for all curves Y corresponding to a dense open subset of the locus $\mathcal{M}(g, H, \mathbf{D})$. This is true with $n = 1$ unless one of the following holds: If $h > 0$ we have $[Aut(X) : H] \leq 2$, and in the case $(0, 4)$ we have H normal in $Aut(X)$ and $[Aut(X) : H] \leq 4$.

Proof : Let $G = \mathrm{Aut}(X)$. Let X be of type $(g, H, \mathbf{D})$ when viewed as H-curve, and of type $(g, G, \mathbf{C})$ when viewed as G-curve.

Note that all maps and spaces in section section3 are defined over $\overline{\mathbb{Q}}$. Hence the condition that X is a generic H-curve means that its corresponding point in $\mathcal{M}_g$ is a generic point of a component of the locus $\mathcal{M}(g, H, \mathbf{D})$. This point lies in the sublocus $\mathcal{M}(g, G, \mathbf{C})$, hence the dimensions are the same: $\delta(g, G, \mathbf{C}) = \delta(g, H, \mathbf{D})$. Now the claim follows from Lemma Lemma4.1. ∎

Lemma 5.2. *(i) Let Σ be a group with a presentation on generators $\sigma_1, ...\sigma_6$ and relations*

$$\sigma_1^2 \ = ... = \ \sigma_6^2 \ = \ 1 \ = \ \sigma_1 \cdots \sigma_6$$

Let Π be the kernel of the map $\Sigma \to \{\pm 1\}$, $\sigma_i \mapsto -1$. Then Π is generated by

$$\begin{array}{lll} \alpha_1 \ = \ \sigma_2\sigma_1 & & \alpha_2 \ = \ \sigma_5\sigma_4 \\ \beta_1 \ = \ \sigma_2\sigma_3 & & \beta_2 \ = \ \sigma_5\sigma_6 \end{array}$$

which gives a presentation for Π with the single relation

$$[\alpha_1, \beta_1] \ [\alpha_2, \beta_2] \ = \ 1$$

We have $\Sigma = \ \Pi \rtimes \langle \sigma_2 \rangle$ where σ_2 acts on Π via

$$\begin{array}{lll} \alpha_1 \mapsto \ \alpha_1^{-1} & & \alpha_2 \mapsto \ (\alpha_2^{-1})^{\alpha_2\beta_2\alpha_1^{-1}\beta_1^{-1}} \\ \beta_1 \mapsto \ \beta_1^{-1} & & \beta_2 \mapsto \ (\beta_2^{-1})^{\alpha_2\beta_2\alpha_1^{-1}\beta_1^{-1}} \end{array} \tag{9}$$

(ii) Let $\Sigma = \ \langle \sigma_1, ...\sigma_5 \rangle$ subject to the relations

$$\sigma_1^2 \ = ... = \ \sigma_4^2 \ = \ 1 \ = \ \sigma_1 \cdots \sigma_5$$

Let Π be the kernel of the map $\Sigma \to \{\pm 1\}$, $\sigma_i \mapsto -1$ for $i = 1, ..., 4$ and $\sigma_5 \mapsto 1$. Then Π is generated by

$$\begin{array}{lll} \alpha \ = \ \sigma_2\sigma_1, & & \gamma_1 \ = \ \sigma_5^{\sigma_4} \\ \beta \ = \ \sigma_2\sigma_3, & & \gamma_2 \ = \ \sigma_5 \end{array}$$

which gives a presentation for Π with the single relation

$$[\alpha, \beta] \ \gamma_1\gamma_2 \ = \ 1$$

Now σ_2 acts on Π via

$$\begin{array}{lll} \alpha \mapsto \ \alpha^{-1} & & \gamma_1 \mapsto \ \gamma_2^{\alpha^{-1}\beta^{-1}} \\ \beta \mapsto \ \beta^{-1} & & \gamma_2 \mapsto \ \gamma_1^{\beta^{-1}\alpha^{-1}} \end{array} \tag{10}$$

(iii) Let $\Sigma = \ \langle \sigma_1, ...\sigma_4 \rangle$ subject to the relations

$$\sigma_1^2 \ = \ \sigma_2^2 \ = \ \sigma_3^2 \ = \ 1 \ = \ \sigma_1 \cdots \sigma_4$$

Let Π be the kernel of the map $\Sigma \to \{\pm 1\}$, $\sigma_i \mapsto -1$ (resp., $\sigma_3 \mapsto (-1, -1)$, the normal subgroup of Σ generated by the conjugates of σ_4). Then Π is generated by

$$\begin{array}{lll} \alpha \ = \ \sigma_2\sigma_1 & \gamma \ = \ \sigma_4^2 & \\ \beta \ = \ \sigma_2\sigma_3 & & \end{array} \quad \textit{resp.,} \quad \begin{array}{lll} \gamma_1 \ = \ \sigma_4 & & \gamma_3 \ = \ \sigma_4^{\sigma_2} \\ \gamma_2 \ = \ \sigma_4^{\sigma_1}, & & \gamma_4 \ = \ \sigma_4^{\sigma_3} \end{array}$$

which gives a presentation for Π with the single relation

$$[\alpha, \beta] \ \gamma \ = \ 1 \qquad \textit{resp.,} \qquad \gamma_1 \cdots \gamma_4 \ = \ 1$$

Now σ_2 acts on Π by inverting α and β (resp., σ_2 switches γ_1, γ_3 and switches γ_2, γ_4, and σ_1 switches γ_1, γ_2 and maps $\gamma_3 \mapsto \gamma_4^{\gamma_1}$, $\gamma_4 \mapsto \gamma_3^{\gamma_2^{-1}}$, and σ_3 switches γ_1, γ_4 and maps $\gamma_2 \mapsto \gamma_3^{\gamma_4}$, $\gamma_3 \mapsto \gamma_2^{\gamma_1^{-1}}$).

Proof : (i) All straightforward computation. One computes that the relation among the α_j, β_j implies the relations among the σ_i, hence the former yields a presentation for Π since $\Sigma = \ \Pi \rtimes \langle \sigma_2 \rangle$. The proof of (ii) and (iii) is similar. ∎

We say that $\alpha_1, \beta_1, ..., \alpha_h, \beta_h, \gamma_1, ..., \gamma_s$ form a **genus g generating system** of H if these elements generate H and satisfy the basic relation (2), where g is given by Riemann-Hurwitz

$$\frac{2 \ (g-1)}{|H|} \quad = \quad 2 \ (h-1) \ + \ \sum_{i=1}^{s} \ \left(1 - \frac{1}{\mathrm{ord}(\gamma_i)}\right) \tag{8}$$

Call a genus g curve X **exceptional** if $X/\mathrm{Aut}(X)$ has genus 0 and exactly three of its points are ramified in X.

Theorem 5.3. *Fix $g \geq 2$. A finite group H is isomorphic to the full automorphism group of a non-exceptional genus g curve if and only if it has a genus g generating system $\alpha_1, \beta_1, ..., \alpha_h, \beta_h, \gamma_1, ..., \gamma_s$ with $(h, s) \neq (0, 3)$ satisfying:*

(a) *If $(h, s) = (2, 0)$ (resp., $(1, 2)$) then there is no automorphism of H acting on the generators via (9) (resp., (10)).*

(b) *If $(h, s) = (1, 1)$ (resp., $(0, 4)$) then there is no automorphism of H inverting α_1 and β_1 (resp., acting on the generators like σ_1, σ_2 or σ_3 in (iii) of the Lemma).*

For the proof we need the following set-up: Let Y be a curve of genus h, and let $q_1, ..., q_s$ be distinct points of Y such that the pair (h, s) is as in (a) or (b). Then there is a Galois cover $Y \to \mathbb{P}^1$ such that the group $A = \mathrm{Aut}(Y/\mathbb{P}^1)$ permutes $q_1, ..., q_s$; further, $|A| = 2$ unless $(h, s) = (0, 4)$ in which case A is a Klein 4 group whose non-trivial elements act as double transpositions on $\{q_1, ..., q_4\}$. If $(Y, q_1, ..., q_s)$ corresponds to a generic point of $\mathcal{M}_{h,s}$ then A is the full stabilizer of $\{q_1, ..., q_s\}$ in $\mathrm{Aut}(Y)$.

Let $\{p_1, ..., p_r\}$ be the subset of $\mathbb{P}^1$ consisting of the points ramified in Y plus the images of $q_1, ..., q_s$. Let t be the identity function on $\mathbb{P}^1$, so that $\mathbb{C}(t)$ is the function field of $\mathbb{P}^1$. View $\mathbb{C}(t)$ as a subfield of $\mathbb{C}(Y)$ naturally. Let $\tilde{\mathcal{M}}$ be a maximal (algebraic) extension of $\mathbb{C}(t)$ unramified outside the places $t = p_1, ..., p_r$, and containing $\mathbb{C}(Y)$. Its Galois group $\tilde{\Sigma}$ is a profinite group generated by elements $\tilde{\sigma}_1, ..., \tilde{\sigma}_r$ subject to the single relation $\tilde{\sigma}_1 \cdots \tilde{\sigma}_r = 1$; here $\tilde{\sigma}_i$ can be chosen to be a distinguished inertia group generator over the place $t = p_i$. Let Ω be the smallest (closed) normal subgroup of $\tilde{\Sigma}$ containing those $\tilde{\sigma}_i^2$ for which there is no q_j over p_i. Then $\Sigma := \tilde{\Sigma}/\Omega$ is the Galois group of $\mathcal{M} := \tilde{\mathcal{M}}^{\Omega}$ over $\mathbb{C}(t)$, and $\mathbb{C}(t) \subset \mathbb{C}(Y) \subset \mathcal{M}$. Let $\Pi = G(\mathcal{M}/\mathbb{C}(Y))$, and let $\sigma_1, ..., \sigma_r$ be the images of the $\tilde{\sigma}_i$ in Σ. Then Σ (resp., Π) is the profinite completion of the group denoted by the same symbol in the Lemma; the cases $(h, s) = (2, 0)$, $(1, 2)$, $(1, 1)$ and $(0, 4)$ correspond to cases (i), (ii) and the two cases in (iii), respectively.

Proof : (of the Theorem) First we prove that the conditon is necessary. So assume $H = \mathrm{Aut}(X)$ for a non-exceptional genus g curve X. Let h be the genus of $Y := X/H$, and $q_1, ..., q_s$ the points of Y ramified in X. It is well-known that then H has a genus g generating system with parameters h, s. So we may assume that (h, s) is as in (a) or (b). Then we are in the set-up discussed before this proof, with $\mathbb{C}(t) \subset \mathbb{C}(Y) \subset \mathbb{C}(X) \subset \mathcal{M}$. The generators of Π from the Lemma yield generators of H via the natural surjection $\Pi = G(\mathcal{M}/\mathbb{C}(Y)) \to G(\mathbb{C}(X)/\mathbb{C}(Y)) = H$. It remains to show that these generators satisfy the condition in (a) and (b).

Let f be the automorphism of Π induced by conjugation action of σ_2 unless $(h, s) = (0, 4)$ when f could be induced by any of $\sigma_1, \sigma_2, \sigma_3$. If H has an automorphism $\bar{f}$ making the following diagram commutative

$$\begin{array}{ccc} \Pi & \xrightarrow{f} & \Pi \\ \downarrow & & \downarrow \\ H & \xrightarrow{\bar{f}} & H \end{array}$$

then the kernel Γ of $\Pi \to H$ is f-invariant. Then the normalizer Δ of Γ in Σ is strictly bigger than Π; thus Δ/Γ is strictly bigger than $\Pi/\Gamma \cong H$. But Δ/Γ acts faithfully on X (since $\Gamma = G(\mathcal{M}/\mathbb{C}(X))$. This contradicts the assumption $H = \mathrm{Aut}(X)$. Thus there is no such commutative diagram, which implies (a) and (b).

Assume now H has a genus g generating system as in the Theorem. It is well-known that then there exists a generic H-curve X of genus g with parameters h, s. If (h, s) is not as in (a) or (b) then $H = \mathrm{Aut}(X)$ by Theorem Theorem5.1 and we are done. Now assume (h, s) is as in (a) or (b). Choose data $(Y, q_1, ..., q_s)$ as in the discussion before this proof, corresponding to a generic point of $\mathcal{M}_{h,s}$. Mapping the generators of Π from the Lemma to the given generators of H defines a

surjection $\Pi \to H$. Let K be the fixed field (in $\mathcal{M}$) of the kernel of this surjection. Then $K = \mathbb{C}(X)$ for a generic H-curve X of genus g. Thus H is normal in $G := \mathrm{Aut}(X)$ by Theorem Theorem5.1. Hence G/H embeds into A, and so $\mathbb{C}(t) \subset K^G \subset \mathbb{C}(Y)$. Thus the map $\Pi \to H$ extends to a map from a group between Π and Σ onto G. Thus if $G \neq H$ then there is an element of G that acts on the given generators of H in the same way that σ_2 (resp., σ_1, σ_2 or σ_3 in case $(h,s) = (0,4)$) acts on the generators of Π (see Lemma). But this is excluded by the condition in (a) resp. (b). Hence $G = H$ and we are done. ∎

Remark 5.4. *The proof shows more: Given g, H and a signature* **e** *(resp., a ramification type $(g, H, \mathbf{E})$), the existence of a generating system of H as in the Theorem with ord$(\gamma_i) = e_i$ (resp., $\gamma_i \in E_i$) is equivalent to the existence of a non-exceptional genus g curve X with Aut$(X) \cong H$ such that the resulting H-curve is of given signature (resp., type).*

Part 3. Classification of automorphism groups

Recall that a group G acts faithfully on a genus g curve if and only if it has a genus g generating system (see (8) above). For g up to 48, all such groups and the signatures of all their genus g generating systems have been listed by Breuer [?Br]. More precisely, for each $g \leq 48$, he produced a list containing all **signature-group pairs** in genus g, i.e., pairs consisting of a group G together with the signature of a genus g generating system of G.

If G acts on X_g then so does each subgroup of G. This shows that Breuer's lists have to be long, and contain some redundancy. Part II of our paper was written in order to eliminate those signature-group pairs that do not yield the full automorphism group of a curve. This can also be done by the results of [?Si] and [?Ri] (as we learned later). It turns out that the larger g gets, the larger is the ratio of entries in Breuer's lists that do occur as full automorphism group in genus g. That can already be seen from the fact (see Lemma Lemma4.1) that if a signature-group pair does not yield the full automorphism group of a curve, then its δ-invariant (dimension of corresponding locus in $\mathcal{M}_g$) is at most 3.

For small g, a relatively large portion of those groups do not occur as full automorphism group in genus g. Among those that do occur, we distinguish those that occur for a particularly simple class of curves: Call a group homocyclic if it is a direct product of isomorphic cyclic groups. A cyclic (resp., homocyclic) cover $X \to \mathbb{P}^1$ is defined to be a Galois cover with cyclic (resp., homocyclic) Galois group C. We call it a **normal cyclic (resp., homocyclic) cover of** $\mathbb{P}^1$ if C is normal in $G = \mathrm{Aut}(X)$. Then $\bar{G} := G/C$ embeds as a finite subgroup of $PSL(2,\mathbb{C})$ and it is easy to write down an equation for X (e.g., in the cyclic case, X has an equation of the form $y^c = f(x)$, where $c = |C|$). To illustrate this, we work out the equations in the genus 3 case. The hyperelliptic case (cyclic with $|C| = 2$) is the simplest case and can be done as in the genus 2 case (see [?ShV]); see also [?Bra] for the cyclic case (in arbitrary characteristic).

6. Genus 3

For $g = 3$, there are 49 signature-group pairs. Applying Theorem **??** and Lemma Lemma4.1 we obtain that 23 of them correspond to G-curves having G as full automorphism group. For each of these 23 signature-group pairs, we find that the corresponding curves form an irreducible locus in $\mathcal{M}_3$, and we compute the equation of a general curve in that locus. There is only two groups (namely C_2 and V_4) that occur with more than one signature, and as soon as $|G| > 2$, then the orbit genus g_0 (genus of X_3/G) is always 0.

Each row in Table 1,2 or 3 corresponds to an irreducible family of curves X_3 with automorphism group G (given in the first column). We let K denote the function field of X_3 over the algebraically closed field k of characteristic 0 (and identify G with $\mathrm{Aut}(K/k)$). We display an equation for K. For almost all values (Zariski-open set) of the parameters in these equations, the equation describes a genus 3 curve with G as full automorphism group. For the exceptional values, the automorphism

group may be larger, or the curve may degenerate. (We do not specify these exceptional values). Most of these equations agree with those found in [?KuKo1], [?KuKo2], but they missed two cases: The hyperelliptic case with $\bar{G} = D_{12}$, and the 3-dimensional hyperelliptic case with $\bar{G} = V_4$. Further, they display an equation for the hyperelliptic case with $\bar{G} = S_4$ which actually belongs to the case $\bar{G} = D_{12}$. Some of the loci they describe with more parameters than the dimension requires. The cases with $g_0 = 1$ and group of order 3 or 4 should be omitted in their list, because the full automorphism group of these curves is larger (as shown by our Theorem **??** above). They note this for two of these cases in the Errata at the end of their paper, but again missed one case.

Now we show how to obtain these equations quickly from the group-theoretic data. Starting point is Breuer's data in genus 3 (see above). We used Part II to eliminate those signature-group pairs that do not yield the full automorphism group of a genus 3 curve. Now we discuss the remaining signature-group pairs.

6.1. **Homocyclic non-hyperelliptic case.** We show how to derive the information in Table 2.

$G = C_4^2 \rtimes S_3$: The fixed field of C_4^2 is of genus 0, call it $k(u)$. The extension $K/k(u)$ is ramified at 3 places of $k(u)$, say $u = 0, 1, \infty$, and the corresponding inertia groups are cyclic of order 4. The fixed fields of these 3 inertia groups extend $k(u)$ with 2 ramified places each, hence they are of genus 0; call them $k(x)$, $k(y)$ and $k(z)$. We may assume $x^4 = u - 1$, $y^4 = u$. Thus $K = k(x, y)$ with $y^4 = u = x^4 + 1$. The equation $y^4 = x(x^2 - 1)$ given by [?KuKo1] is also correct. (Indeed, the four roots of $x^4 + 1$ have j-invariant 1728, same as ∞ plus the 3 roots of $x(x^2 - 1)$).

$G = (48, 33)$: Here G is a central extension of C_4 by quotient A_4. Let $H \cong C_3$ be a Sylow 3-subgroup of G. The fixed field of H is a $k(x)$, and $K/k(x)$ is ramified at 5 places of $k(x)$. Let $h_1, ..., h_5$ be the corresponding distinguished inertia group generators in H. Their product is 1, hence we may assume $h_1 = ... = h_4 = h_5^{-1}$. Take the place with generator h_5 to be $x = \infty$, and the other 4 ramified places to be the roots of $f(x)$. Then $K = k(x, y)$ with $y^3 = f(x)$. Further, the C_4 acts on $k(x)$ permuting the 4 roots of $f(x)$. We can choose x such that C_4 acts on $k(x)$ by the maps $x \mapsto \zeta x$, $\zeta^4 = 1$. Then $f(x) = x^4 - b$. We can get each $b \neq 0$ by further normalization of x. This agrees with [?KuKo1].

$G = (16, 13)$: Here G is a central extension of $H \cong C_4$ by quotient V_4. The fixed field of H is a $k(x)$, and $K/k(x)$ is ramified at 4 places $p_1, ..., p_4$ of $k(x)$. Let $h_1, ..., h_4$ be the corresponding inertia elements in H. The Klein 4 group G/H permutes $p_1, ..., p_4$. If this action were not transitive, then $p_1, ..., p_4$ could not be 4 general points in $\mathbb{P}^1$, contradicting the fact that the covers under consideration form a 1-dimensional family. Thus G/H permutes $p_1, ..., p_4$ transitively, hence permutes $h_1, ..., h_4$ transitively (by conjugation). Thus $h_1 = ... = h_4$ since H is central in G. (Alternatively, this can be shown by the algorithm before Lemma Lemma4.1). Hence $K = k(x, y)$ with f of degree 4. The generic curve of this type has automorphism group of order 16, because there is always a Klein 4 group in $\mathrm{Aut}(k(x))$ permuting the 4 roots of f.

$G = C_9, C_6$ or C_3: Let H be the subgroup of G of order 3. The fixed field of H is a $k(x)$, and $K/k(x)$ is ramified at 5 places $p_1, ..., p_5$ of $k(x)$. Let $h_1, ..., h_5$ be the corresponding inertia elements. They have product 1, hence we may assume $h_1 = ... = h_4 = h_5^{-1}$. The group G/H permutes $p_1, ..., p_5$, fixing p_5. Take $p_1, ..., p_5$ to be the places $x = 0, 1, s, t, \infty$, respectively. Then $K = k(x, y)$ with $y^3 = x(x - 1)(x - s)(x - t)$. Let α be the automorphism of $k(x)$ induced by a generator of G/H. If it has order 3 then we may assume that $x = 0$ is the other fixed point of α (besides $x = \infty$); then $\alpha(x) = \zeta_3 x$ and the equation becomes $y^3 = x(x^3 - 1)$. Now assume α has order 2. We may assume it switches 0 and 1. Then $\alpha(t) = 1 - t$, again as claimed.

$G = S_4, D_8$ or V_4: Let H a normal Klein 4-group in G. The fixed field of H is a $k(w)$, and $K/k(w)$ is ramified at 6 places $p_1, ..., p_6$ of $k(w)$. Let $h_1, ..., h_6$ be the corresponding inertia elements. If they don't comprise all involutions in H then there is $h \in H$ that occurs 4 times among the h_i;

then K^h extends $k(w)$ with 2 branch points, hence K^h has genus 0 and so K is hyperelliptic. We excluded that case. Thus we may assume $h_1 = h_2$, $h_3 = h_4$, $h_5 = h_6$ and $\{h_1, h_3, h_5\} = H \setminus \{1\}$.

Let u, v be independent transcendentals over k with $w = u/v$. The places of $k(w)$ correspond to homogeneous degree 1 polynomials in u, v, modulo scalar multiples. Let P, Q, R be homogeneous degree 2 polynomials in $k[u, v]$ such that the degree 1 factors of P (resp., Q, resp., R) correspond to p_1, p_2 (resp., p_3, p_4, resp., p_5, p_6). The quotients P/Q, R/Q, P/R are naturally elements of $k(w)$. Then $K^{h_3} = k(w, x)$ with $x^2 = P/Q$, and $K^{h_1} = k(w, y)$ with $y^2 = R/Q$, and $K^{h_2} = k(w, z)$ with $z^2 = P/R$. Thus $K = k(w, x, y)$. We claim $K = k(x, y)$. Assume wrong. Then $[K : k(x, y)] = 2$ since w satisfies the equation $P(w, 1) - x^2 Q(w, 1) = 0$ over $k(x, y)$. Let g_0 be the generator of $G(K/k(x, y))$. Then g_0 is not in H, since none of the h_i fixes both x and y. But each h_i maps x (resp., y) to $\pm x$ (resp., $\pm y$), hence H leaves $k(x, y)$ invariant and thus centralizes g_0. Hence $< H, g_0 > \cong C_2^3$, contradiction (since G contains no such subgroup). This proves $K = k(x, y)$.

The 6 homogeneous polynomials $P^2, Q^2, R^2, PQ, PR, QR$ of degree 4 are linearly dependent in $k[u, v]$. This gives a non-trivial relation

$$aPR \ + \ bPQ \ + \ cQR \ + \ dP^2 \ + \ eQ^2 \ + \ fR^2 \ = \ 0$$

If $f = 0$ then dividing by Q^2 gives an equation for y over $k(x)$ of y-degree ≤ 2 and so K is hyperelliptic or of genus 0 – contradiction. Thus f (and similarly d, e) are non-zero. Replacing P, Q, R by suitable scalar multiples we get a relation

$$P^2 \ + \ Q^2 \ + \ R^2 \ + \ aPR \ + \ bPQ \ + \ cQR \ = \ 0$$

with $a, b, c \in k$. Dividing by Q^2 gives

$$x^4 \ + \ 1 \ + \ y^4 \ + \ ax^2y^2 \ + \ bx^2 \ + \ cy^2 \ = \ 0$$

This settles the case $G = V_4$.

Suppose now $G = D_8$. Then G/H induces an involutory automorphism α of $k(w)$ switching (without loss) the sets $\{p_1, p_2\}$ and $\{p_5, p_6\}$. We may assume $\alpha(w) = 1/w$. Extend it to $k(u, v)$ by $\alpha(u) = v$, $\alpha(v) = u$. Then α fixes Q and switches P, R up to scalar multiples. Since $\alpha^2 = \mathrm{id}$, we have $\alpha(Q) = \pm Q$. Applying α to the relation between P, Q, R yields that $\alpha(P) = \pm R$. Replacing R by $-R$ if necessary we may assume $\alpha(P) = R$ (resp., $\alpha(P) = -R$) if $\alpha(Q) = Q$ (resp., $\alpha(Q) = -Q$). Then $b = c$. Finally, suppose $G = S_4$. Then there is another automorphism like α, fixing R and switching P, Q up to scalar multiples. Replacing Q by $-Q$ if necessary yields $a = b = c$.

6.2. **Hyperelliptic case.** Here we derive the information in Table 3.

In the hyperelliptic case, G has a central subgroup C of order 2 whose fixed field in K is a $k(x)$. There are 8 places of $k(x)$ that ramify in K, say $x = a_1, ..., a_8$. Then $\bar{G} = G/C$ embeds into $PGL(2, k)$ as a subgroup permuting $a_1, ..., a_8 \in \mathbb{P}^1$. Let σ_m (resp., τ) be the element of $PGL(2, k)$ mapping x to $\zeta_m x$ (resp., $1/x$), where ζ_m is a primitive m-th root of 1 in k. Then each subgroup of $PGL(2, k)$ isomorphic to C_m resp., D_{2m} is conjugate to $< \sigma_m >$ resp., $< \sigma_m, \tau >$. We identify C_m (resp., D_m) with $< \sigma_m >$ (resp., $< \sigma_m, \tau >$). In particular, $D_2 = V_4 =< \sigma_2, \tau >$.

$\bar{G} = S_4$: Let $i \in k$ with $i^2 = -1$. We may assume $\bar{G}$ contains D_2 and $\rho : x \mapsto (x + i)/(x - i)$. ($\rho$ is of order 3, permuting the 6 fixed points of the involutions in V_4). The two fixed points of ρ are the roots of $X^2 - (i + 1)X - i = 0$. The group $< V_4, \rho > \cong A_4$ has exactly one orbit of length 8 on $\mathbb{P}^1$, containing those fixed points of ρ. Hence the orbit consists of the roots of $(X^2-(i+1)X-i)(X^2+(i+1)X-i)(-iX^2-(i+1)X+1)(-iX^2+(i+1)X+1) \ = \ X^8+14X^4+1$.

$\bar{G} = D_{16}, D_{12}, C_7$: In the dihedral case, $\bar{G}$ leaves exactly one set of length 8 in $\mathbb{P}^1$ invariant, consisting of the roots of $X^8 - 1$, resp., the roots of $X^6 - 1$ plus 0 and ∞. In the third case, we may assume $a_1 = 1$ by a coordinate change; then $a_1, ..., a_8$ comprise the roots of $X^7 - 1$ plus either 0 or ∞. (The latter two choices are conjugate under the map τ). The case $\bar{G} = D_{12}$ is missing in [?KuKo1] and the corresponding equation $y^2 = x(x^6 - 1)$ is erroneously associated with the case $\bar{G} = S_4$.

$\bar{G} = D_8, S_3$: Here $a_1, ..., a_8$ comprise a regular orbit of D_8 resp., 0 and ∞ plus a regular orbit. Thus in the first case, $a_1, ..., a_8$ are the roots of $(X^4 - t)(X^4 - 1/t) = X^8 - (t + 1/t)X^4 + 1$. The other case is similar.

$\bar{G} = V_4$: Here $a_1, ..., a_8$ either comprise two regular orbits of $\bar{G}$, or a regular orbit plus two of three orbits of length 2. The orbits of length 2 are all conjugate under S_4, so we may take $\{0, \infty\}$ and $\{\pm 1\}$ as the two orbits of length 2. Now the claim follows as in the previous case.

$\bar{G} = C_2$: Here $a_1, ..., a_8$ either comprise 4 regular orbits, or 0 and ∞ plus 3 regular orbits. We may assume $a_1 = 1$ by a coordinate change.

6.3. **Remaining cases, collected in Table 1.** Here we take X_3 in its canonical embedding in the projective plane associated with the dual of the 3-dimensional (faithful) G-module Ω of holomorphic differential forms on X_3. Let x, y, z be coordinates of this module Ω^*. Let $f(x, y, z) = 0$ be the equation of X_3 in these coordinates. Then f spans a 1-dimensional fixed space for the representation of G on the space of quartic homogeneous polynomials in x, y, z.

To identify the representation of G on Ω, we use that the space of fixed points of G on Ω has dimension equal to the genus g_0 of X_3/G (see [?Br]); this fixed point space can be identified with $\Omega(X_3/G))$.

$G = L_3(2)$: This is the well-known Hurwitz group of order 168. The $(2, 3, 7)$ triple is weakly-rigid in G, so there is exactly one associated curve found already by Klein.

$G = S_3, g_0 = 0$: The group S_3 has only one faithful 3-dimensional representation with trivial fixed space: It is the natural permutation representation tensored with the 1-dimensional sign-representation. Thus G acts in the projective plane associated with Ω^* just by permuting the homogeneous coordinates x, y, z (for suitable choice of such coordinates). Thus the quartic polynomial $f(x, y, z)$ defining X_3 is either symmetric in x, y, z, or has the property that it is invariant under a cyclic permutation of x, y, z, but changes sign under a transposition. The only possibility (up to scalar multiples) for the latter case is $f = x^3y + y^3z + z^3x - y^3x - z^3y - x^3z$, but this polynomial is not irreducible (product of linear factors). Hence f is symmetric, thus is of the form

$$f = a(x^4+y^4+z^4)+b(x^2y^2+x^2z^2+y^2z^2)+c(x^2yz+y^2xz+z^2xy)+d(x^3y+x^3z+y^3x+y^3z+z^3x+z^3y).$$

The centralizer of S_3 in $GL_3(\mathbb{C})$ consists of the matrices $tI + sJ$, where I (resp., J) is the identity matrix (resp., the all 1's matrix). By a coordinate change with such a matrix we can make $d = 0$, but not uniquely: Generically, there is always 4 triples (a, b, c) (up to scalar multiples) that give the same curve (up to isomorphism).

$G = C_2, g_0 = 1$: We may assume that the involution in G acts as $x \mapsto -x, y \mapsto y, z \mapsto z$. Thus f either contains only even powers of x (i.e., f is fixed by G) or f contains only odd powers of x (i.e., f is mapped to $-f$ by the involution in G). In the latter case, f is reducible (product of x and a genus 1 equation). Thus f contains only even powers of x, hence

$$f \quad = \quad a_0x^4 \; + \; x^2 \; (a_1y^2 + a_2yz + a_3z^2) \; + \; a_4y^4 + a_5y^3z + a_6y^2z^2 + a_7yz^3 + a_8z^4$$

By a linear change of the variables y, z we can make $a_1 = 1$ and $a_2 = 0$. (If $a_1 = a_2 = a_3 = 0$ then X_3 has an automorphism of order 4, namely multiplying x by ζ_4, so we don't consider this case here). Further we can replace x and z by scalar multiples to get the normalizations from Table 1. (If either $a_0 = 0$ or $a_7 = a_8 = 0$ then the curve is hyperelliptic).

TABLE 1. $\mathrm{Aut}(X_3)$ for X_3 not a normal homocyclic cover of $\mathbb{P}^1$

G	signature	orbit genus	proj. equation	Group ID
$L_3(2)$	$(2,3,7)$	0	$x^3y + y^3z + z^3x = 0$	(168,42)
S_3	$(2,2,2,2,3)$	0	$a(x^4+y^4+z^4) + b(x^2y^2+x^2z^2+y^2z^2)+$ $c(x^2yz + y^2xz + z^2xy) = 0$	(6,1)
C_2	$(2,2,2,2)$	1	$x^4 + x^2(y^2+az^2) + by^4 + cy^3z + dy^2z^2 + eyz^3 + gz^4 = 0$ either $e = 1$ or $g = 1$	(2,1)

TABLE 2. $\mathrm{Aut}(X_3)$ for normal homocyclic covers of $\mathbb{P}^1$, not hyperelliptic

G	C	G/C	signature	equation	Group ID
V_4	V_4	$\{1\}$	(2^6)	$x^4+y^4+ax^2y^2+bx^2+cy^2+1=0$	(4,2)
D_8	V_4	C_2	(2^5)	take $b=c$	(8,3)
S_4	V_4	S_3	$(2,2,2,3)$	take $a=b=c$	(24,12)
$C_4^2 \rtimes S_3$	V_4	S_4	$(2,3,8)$	take $a=b=c=0$ or $y^4 = x(x^2-1)$	(96,64)
16	C_4	V_4	$(2,2,2,4)$	$y^4 = x(x-1)(x-t)$	(16,13)
48	C_4	A_4	$(2,3,12)$	$y^4 = x^3 - 1$	(48,33)
C_3	C_3	$\{1\}$	(3^5)	$y^3 = x(x-1)(x-s)(x-t)$	(3,1)
C_6	C_3	C_2	$(2,3,3,6)$	take $s = 1-t$	(6,2)
C_9	C_3	C_3	$(3,9,9)$	$y^3 = x(x^3-1)$	(9,1)

TABLE 3. $\mathrm{Aut}(X_3)$ for hyperelliptic X_3

G	G/C	signature	dim. of locus	equation $y^2 =$	Group ID
48	S_4	$(2,4,6)$	0	$x^8 + 14x^2 + 1$	$(48,48)$
32	D_{16}	$(2,4,8)$	0	$x^8 - 1$	$(32,9)$
24	D_{12}	$(2,4,12)$	0	$x(x^6-1)$	$(24,5)$
C_{14}	C_7	$(2,7,14)$	0	$x^7 - 1$	$(14,2)$
16	D_8	$(2,2,2,4)$	1	$x^8 + ax^4 + 1$	$(16,11)$
D_{12}	S_3	$(2,2,2,6)$	1	$x(x^6 + ax^3 + 1)$	$(12,4)$
$C_2 \times C_4$	V_4	$(2,2,4,4)$	1	$x(x^2-1)(x^4+ax^2+1)$	$(8,2)$
C_2^3	V_4	(2^5)	2	$(x^4+ax^2+1)(x^4+bx^2+1)$	$(8,5)$
C_4	C_2	$(2,2,2,4,4)$	2	$x(x^2-1)(x^4+ax^2+b)$	$(4,1)$
V_4	C_2	(2^6)	3	$(x^2-1)(x^6+ax^4+bx^2+c)$	$(4,2)$
C_2	$\{1\}$	(2^8)	5	$x(x-1)(x^5+ax^4+bx^3+cx^2+dx+e)$	$(2,1)$

7. Curves of genus ≤ 10 with large automorphism group

In the genus 3 case, we were able to write out explicit equations for the curves with any given automorphism group. This yields an explicit description of the corresponding loci in $\mathcal{M}_3$. For higher genus, we cannot expect to obtain explicit equations. Still, computational group theory allows us to determine the dimensions and number of components of these loci. This uses the general set-up of section 3.

The number of signature-group pairs grows quickly with the genus. E.g., in genus 10 there are already 174 signature-group pairs with $g_0 = 0$, and most of them yield the full automorphism group of a genus 10 curve. So it would not be feasible to display all automorphism groups up to genus 10. Therefore, we only display the **large** groups $\mathrm{Aut}(X_g)$ (in the sense of section 1.3), see Table 4. Surprisingly, their number remains relatively small. They comprise the most interesting groups in each genus, and we avoid listing the many group-signature pairs with group of order 2, 3 etc.

7.1. **The general set-up.** We return to the set-up of section 3. Let $\mathbf{c} = (c_1, ..., c_r)$ be the signature of a genus g generating system of G. Let $\mathcal{H}(g, G, \mathbf{c})$ be Hurwitz space parameterizing equivalence classes of G-curves of genus g and signature $\mathbf{c}$; i.e., $\mathcal{H}(g, G, \mathbf{c})$ is the (disjoint) union of the spaces $\mathcal{H}(g, G, \mathbf{C})$ with $\mathbf{C}$ of signature $\mathbf{c}$ (see section 3). The map

$$\Phi : \quad \mathcal{H}(g, G, \mathbf{c}) \quad \to \quad \mathcal{M}_g$$

forgetting the G-action is a finite morphism. Let $\mathcal{M}(g, G, \mathbf{c})$ be its image (the locus of genus g curves admitting a G-action of signature $\mathbf{c}$). All components of $\mathcal{M}(g, G, \mathbf{c})$ have dimension $\delta = \delta(g, G, \mathbf{c})$ (by Lemma 3.1). In particular, if G is large then $g_0 = 0$ and $r = 3, 4$, hence δ is 0 or 1.

Define $\mathcal{H}^*(g, G, \mathbf{c})$ (resp., $\mathcal{H}^*(g, G, \mathbf{C})$ with $\mathbf{C}$ a tuple of conjugacy classes, see section 3) to be the union of all components C of $\mathcal{H}(g, G, \mathbf{c})$ (resp., $\mathcal{H}(g, G, \mathbf{C})$) with the following property: There is at least one point on C such that the associated G-curve has G as full automorphism group. Let $\mathcal{M}^*(g, G, \mathbf{c})$ (resp., $\mathcal{M}^*(g, G, \mathbf{C})$) be the Φ-image of the corresponding space $\mathcal{H}^*(...)$. Then Φ is generically injective on $\mathcal{M}^*(g, G, \mathbf{c})$, and so the spaces $\mathcal{M}^*(g, G, \mathbf{c})$ and $\mathcal{H}^*(g, G, \mathbf{c})$ have the same number of components. In the case $g_0 = 0$, the spaces $\mathcal{H}(g, G, \mathbf{c})$ coincide with the Hurwitz spaces studied in [?FV], [?V], [?Buch]. Thus the components of $\mathcal{H}(g, G, \mathbf{c})$ correspond to the braid orbits of (genus 0) generating systems of G of signature $\mathbf{c}$, taken modulo $\mathrm{Aut}(G)$. These braid orbits can be computed with the BRAID package [?MSV].

7.2. **The table of large automorphism groups up to genus 10.** In Table 4 we list all group-signature pairs $(G, \mathbf{c})$ of genus g, where $4 \leq g \leq 10$, with the following property: There exists a G-curve X_g of genus g and signature $\mathbf{c}$ such that G is the full automorphism group of X_g and G is large; i.e,

$$|G| > 4(g - 1).$$

More precisely, the rows of Table 4 correspond to the components of the loci $\mathcal{M}^*(g, G, \mathbf{c})$ associated with these parameters. It turns out that these loci are mostly irreducible, with only 6 exceptions listed in Table 5. In these exceptional cases, they have two components, and these components are all of the form $M^*(g, G, \mathbf{C})$, with $\mathbf{C}$ a tuple of conjugacy classes (see 7.1). Thus the spaces $M^*(g, G, \mathbf{C})$ are always irreducible in the situation of Table 4, and they correspond bijectively to the rows of Table 4. In particular, duplicate rows occur iff the corresponding locus $\mathcal{M}^*(g, G, \mathbf{c})$ is reducible. The group G is identified via its ID from the Small Group Library. In the last column of Table 4, we also indicate the inclusion relations between components of dimension 0 and 1. They can be computed by the algorithm given before Lemma Lemma4.1.

TABLE 4. Components of the Hurwitz loci $\mathcal{M}^*(g, G, \mathbf{C})$ for large G

#	Group ID	signature	contains	#	Group ID	signature	contains
Genus 4, $\delta = 0$							
1	(120,34)	$(2, 4, 5)$		**2**	(72,42)	$(2, 3, 12)$	
3	(72,40)	$(2, 4, 6)$		**4**	(40,8)	$(2, 4, 10)$	
5	(36,12)	$(2, 6, 6)$		**6**	(32,19)	$(2, 4, 16)$	
7	(24,3)	$(3, 4, 6)$		**8**	(18,2)	$(2, 9, 18)$	
9	(15,1)	$(3, 5, 15)$					
Genus 4, $\delta = 1$							
10	(36,10)	$(2, 2, 2, 3)$	3	**11**	(24,12)	$(2, 2, 2, 4)$	1, 2
12	(20,4)	$(2, 2, 2, 5)$	4	**13**	(18,3)	$(2, 2, 3, 3)$	2, 5
14	(16,7)	$(2, 2, 2, 8)$	6				
Genus 5, $\delta = 0$							
1	(192,181)	$(2, 3, 8)$		**2**	(160,234)	$(2, 4, 5)$	
3	(120,35)	$(2, 3, 10)$		**4**	(96,195)	$(2, 4, 6)$	
5	(64,32)	$(2, 4, 8)$		**6**	(48,14)	$(2, 4, 12)$	
7	(48,30)	$(3, 4, 4)$		**8**	(40,5)	$(2, 4, 20)$	
9	(30,2)	$(2, 6, 15)$		**10**	(22,2)	$(2, 11, 22)$	
Genus 5, $\delta = 1$							
11	(48,48)	$(2, 2, 2, 3)$	1, 4	**12**	(32,43)	$(2, 2, 2, 4)$	
13	(32,28)	$(2, 2, 2, 4)$	1	**14**	(32,27)	$(2, 2, 2, 4)$	2, 4, 5
15	(24,14)	$(2, 2, 2, 6)$	6	**16**	(24,8)	$(2, 2, 2, 6)$	4
17	(24,13)	$(2, 2, 3, 3)$	3, 7	**18**	(20,4)	$(2, 2, 2, 10)$	3, 8
Genus 6, $\delta = 0$							
1	(150,5)	$(2, 3, 10)$		**2**	(120,34)	$(2, 4, 6)$	
3	(72,15)	$(2, 4, 9)$		**4**	(56,7)	$(2, 4, 14)$	
5	(48,6)	$(2, 4, 24)$		**6**	(48,29)	$(2, 6, 8)$	
7	(48,15)	$(2, 6, 8)$		**8**	(39,1)	$(3, 3, 13)$	
9	(30,1)	$(2, 10, 15)$		**10**	(26,2)	$(2, 13, 26)$	
11	(21,2)	$(3, 7, 21)$					
Genus 6, $\delta = 1$							
12	(60,5)	$(2, 2, 2, 3)$	2	**13**	(28,3)	$(2, 2, 2, 7)$	4
14	(24,12)	$(2, 2, 3, 4)$	2	**15**	(24,8)	$(2, 2, 3, 4)$	3
16	(24,6)	$(2, 2, 2, 12)$	5	**17**	(24,6)	$(2, 2, 3, 4)$	7

TABLE 4. (Cont.)

#	Group ID	signature	contains	#	Group ID	signature	contains
Genus 7, $\delta = 0$							
1	(504,156)	$(2,3,7)$		**2**	(144,127)	$(2,3,12)$	
3	(64,41)	$(2,4,16)$		**4**	(64,38)	$(2,4,16)$	
5	(56,4)	$(2,4,28)$		**6**	(54,6)	$(2,6,9)$	
7	(54,6)	$(2,6,9)$		**8**	(54,3)	$(2,6,9)$	
9	(48,32)	$(3,4,6)$		**10**	(42,4)	$(2,6,21)$	
11	(32,11)	$(4,4,8)$		**12**	(32,10)	$(4,4,8)$	
13	(30,4)	$(2,15,30)$					
Genus 7, $\delta = 1$							
14	(48,48)	$(2,2,2,4)$		**15**	(48,41)	$(2,2,2,4)$	2
16	(48,38)	$(2,2,2,4)$		**17**	(36,10)	$(2,2,2,6)$	
18	(32,43)	$(2,2,2,8)$		**19**	(32,42)	$(2,2,2,8)$	3
20	(32,39)	$(2,2,2,8)$	4	**21**	(28,3)	$(2,2,2,14)$	5
Genus 8, $\delta = 0$							
1	(336,208)	$(2,3,8)$		**2**	(336,208)	$(2,3,8)$	
3	(84,7)	$(2,6,6)$		**4**	(84,7)	$(2,6,6)$	
5	(72,8)	$(2,4,18)$		**6**	(64,53)	$(2,4,32)$	
7	(60,8)	$(2,6,10)$		**8**	(48,25)	$(2,6,24)$	
9	(48,17)	$(2,8,12)$		**10**	(48,28)	$(3,4,8)$	
11	(40,10)	$(2,10,20)$		**12**	(34,2)	$(2,17,34)$	
Genus 8, $\delta = 1$							
13	(42,1)	$(2,2,3,3)$	1, 2, 3, 4	**14**	(36,4)	$(2,2,2,9)$	5
15	(32,18)	$(2,2,2,16)$	6	**16**	(30,3)	$(2,2,3,5)$	7
Genus 9, $\delta = 0$							
1	(320,1582)	$(2,4,5)$		**2**	(192,194)	$(2,3,12)$	
3	(192,990)	$(2,4,6)$		**4**	(192,955)	$(2,4,6)$	
5	(128,138)	$(2,4,8)$		**6**	(128,136)	$(2,4,8)$	
7	(128,134)	$(2,4,8)$		**8**	(128,75)	$(2,4,8)$	
9	(120,35)	$(2,5,6)$		**10**	(120,34)	$(2,5,6)$	
11	(96,187)	$(2,4,12)$		**12**	(96,186)	$(2,4,12)$	
13	(96,13)	$(2,4,12)$		**14**	(80,14)	$(2,4,20)$	
15	(72,5)	$(2,4,36)$		**16**	(64,6)	$(2,8,8)$	
17	(57,1)	$(3,3,19)$		**18**	(48,5)	$(2,8,24)$	
19	(48,4)	$(2,8,24)$		**20**	(48,30)	$(4,4,6)$	
21	(42,3)	$(2,14,21)$		**22**	(40,12)	$(4,4,10)$	
23	(38,2)	$(2,19,38)$					

TABLE 4. (Cont.)

#	Group ID	signature	contains	#	Group ID	signature	contains
Genus 9, $\delta = 1$							
24	(96,193)	$(2,2,2,3)$	3	**25**	(96,227)	$(2,2,2,3)$	4
26	(64,190)	$(2,2,2,4)$		**27**	(64,177)	$(2,2,2,4)$	5
28	(64,140)	$(2,2,2,4)$		**29**	(64,138)	$(2,2,2,4)$	4
30	(64,135)	$(2,2,2,4)$	1, 3, 6	**31**	(64,134)	$(2,2,2,4)$	7
32	(64,128)	$(2,2,2,4)$		**33**	(64,73)	$(2,2,2,4)$	2, 8
34	(48,43)	$(2,2,2,6)$	13	**35**	(48,38)	$(2,2,2,6)$	
36	(48,15)	$(2,2,2,6)$	3	**37**	(48,48)	$(2,2,2,6)$	4, 12
38	(48,48)	$(2,2,2,6)$	11	**39**	(40,13)	$(2,2,2,10)$	14
40	(40,8)	$(2,2,2,10)$		**41**	(36,4)	$(2,2,2,18)$	15
Genus 10, $\delta = 0$							
1	(432,734)	$(2,3,8)$		**2**	(432,734)	$(2,3,8)$	
3	(360,118)	$(2,4,5)$		**4**	(324,160)	$(2,3,9)$	
5	(216,92)	$(2,3,12)$		**6**	(216,158)	$(2,4,6)$	
7	(216,87)	$(2,4,6)$		**8**	(216,153)	$(3,3,4)$	
9	(180,19)	$(2,3,15)$		**10**	(168,42)	$(2,4,7)$	
11	(162,14)	$(2,3,18)$		**12**	(144,122)	$(2,3,24)$	
13	(108,25)	$(2,6,6)$		**14**	(108,15)	$(2,4,12)$	
15	(88,7)	$(2,4,22)$		**16**	(80,6)	$(2,4,40)$	
17	(72,28)	$(2,6,12)$		**18**	(72,23)	$(2,6,12)$	
19	(72,42)	$(3,4,6)$		**20**	(63,3)	$(3,3,21)$	
21	(60,10)	$(2,6,30)$		**22**	(42,6)	$(2,21,42)$	
23	(42,2)	$(3,6,14)$		**24**	(42,2)	$(3,6,14)$	
Genus 10, $\delta = 1$							
25	(108,40)	$(2,2,2,3)$	4, 6	**26**	(108,17)	$(2,2,2,3)$	1, 2, 7
27	(72,43)	$(2,2,2,4)$	5	**28**	(72,40)	$(2,2,2,4)$	1, 2, 6
29	(72,15)	$(2,2,2,4)$		**30**	(60,5)	$(2,2,2,5)$	3, 9
31	(54,8)	$(2,2,2,6)$	11, 14	**32**	(54,5)	$(2,2,3,3)$	5, 8, 13
33	(48,29)	$(2,2,2,8)$	1, 2, 12	**34**	(44,3)	$(2,2,2,11)$	15
35	(40,6)	$(2,2,2,20)$	16				

TABLE 5. Reducible Hurwitz loci $\mathcal{M}^*(g, G, \mathbf{c})$

genus	Group ID	signature	components of locus
7	(54,6)	(2,6,9)	6,7
8	(336,208)	(2,3,8)	1,2
8	(84,7)	(2,6,6)	3,4
9	(48,48)	(2,2,2,6)	37,38
10	(432,734)	(2,3,8)	1,2
10	(42,2)	(3,6,14)	23,24

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