Vojta's conjecture on weighted projective varieties

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Abstract

We formulate Vojta's conjecture for smooth weighted projective varieties, weighted multiplier ideal sheaves, and weighted log pairs and prove that all three versions of the conjecture are equivalent. In the process, we introduce generalized weighted general common divisors and express them as heights of weighted projective spaces blown-up relative to an exceptional divisor. Furthermore, we prove that assuming Vojta's conjecture for weighted projective varieties one can bound the $\mathbf{log\, h_{wgcd}}$ for any subvariety of codimension $\geq \mathbf{2}$ and a finite set of places $\boldsymbol{S}$. An analogue result is proved for weighted homogeneous polynomials with integer coefficients.

Keywords: Weighted Vojta's Conjecture, Weighted generalized common divisor, Weighted blow-up

1 Introduction

The theory of weighted local and global heights for weighted projective varieties and closed subvarieties was introduced in [13]. Weighted heights provide an efficient way of dealing with rational points on weighted varieties compared to the classical approach of considering their projective image under the Veronese map. Some of the natural questions on this new approach is whether the main conjectures of arithmetic geometry can be formulated for weighted varieties and weighted heights. This paper accomplishes

exactly that for one of the most important conjectures of arithmetic geometry, that of Vojta's conjecture.

We generalize the generalized logarithmic greatest common divisors to generalized logarithmic weighted greatest common divisors and express them as a weighted height of a blow-up relative to a weighted exceptional divisor. Moreover, we are able to state Vojta's conjecture for smooth weighted projective varieties in terms of weighted heights.

This paper is organized as follows: In Sec. 2, we give a quick summary of some of the basics of weighted projective spaces $\mathbb{P}^n_{\mathfrak{q},k}$, their weighted heights, and the concept of rational points on weighted varieties. In the last part of Sec. 2, we provide an example of a weighted hypersurface having no rational points of weighted height ≤ 2 in $\mathbb{P}^4_{(2,4,6,10),\mathbb{Q}}$. This corresponds to the locus $\mathcal{L}_2$ of genus two curves with extra involutions in weighted projective space in $\mathbb{P}^4_{(2,4,6,10),\mathbb{Q}}$; see [14] for more details on this topic. The analogue statement in terms of projective varieties is that the projective image $\phi_{60}(\mathcal{L}_2)$, under the Veronese map ϕ_{60}, has no rational points of projective height $\leq 2^{60}$. This is to illustrate that the bound on the height changes so dramatically that it is well worth it to consider arithmetic properties of the weighted variety instead of its projective counterpart.

In Sec. 3, we review the Vojta's conjecture on algebraic points on projective varieties; see Conj. 1, We state the conjecture using a correction term involving a multiplier ideal sheaf instead of using the normal crossing divisors; see Conj. 2, and summarize [19] on Vojta's conjecture for log pairs. This makes it possible to drop the condition that the variety be smooth in the statement of the conjecture. Vojta's conjecture for log pairs is stated in Conj. 3.

In Sec. 4, we investigate whether it is possible to have analog statements for weighted varieties and weighted heights. In Conj. 4, we state Vojta's conjecture for $\mathcal{X}$ a smooth weighted projective variety, $K_{\mathcal{X}}$ a canonical divisor, $\mathcal{A}$ an ample divisor and D a normal crossings divisor on $\mathcal{X}$, all defined over $\mathbb{Q}$. An analogue of Conj. 2, for weighted projective varieties is stated in Conj. 5. The terminology and theory for weighted log pairs is developed in this section, so we are able to state Vojta's conjecture for weighted log pairs in Conj. 6. Finally, in Cor. 1, we prove that Conj. 4, Conj. 5, and Conj. 6 are equivalent.

In Sec. 5, we extend the concept of the *generalized greatest common divisor* to that of *generalized weighted greatest common divisor*. Furthermore, we prove that generalized logarithmic weighted greatest common divisor is equal to weighted height of points on a blow-up of $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ with respect to the exceptional divisor of the blow-up.

More precisely, we prove (cf. Thm. 1) that for $\mathcal{X}$ a smooth weighted variety in $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$, $\mathcal{A}$ an ample divisor on $\mathcal{X}$, $\mathcal{Y} \subset \mathcal{X}$ a smooth subvariety of codimension $r \geq 2$, and $-K_{\mathcal{X}}$ a normal crossing divisor whose support does not intersect $\mathcal{Y}$, assuming Conj. 4, for every finite set of places S and every $0 < \varepsilon < r - 1$ there is a proper closed subvariety $\mathcal{Z} = \mathcal{Z}(\varepsilon, \mathcal{X}, \mathcal{Y}, \mathcal{A}, S)$ of $\mathcal{X}$, and constants

$$C_\varepsilon = C_\varepsilon(\mathcal{X}, \mathcal{Y}, \mathcal{A}, S) \quad \textit{and} \quad \delta_\varepsilon = \delta_\varepsilon(\mathcal{X}, \mathcal{Y}, \mathcal{A}),$$

such that for all $P \in (\mathcal{X} \setminus \mathcal{Z})(\mathbb{Q})$ one has

$$\log \mathrm{h}_{\mathrm{wgcd}}(P;\mathcal{Y}) \leq \varepsilon\, \mathfrak{s}_{\mathcal{X},\mathcal{A}}(P) + \frac{1}{r-1+\delta_\varepsilon}\, \mathfrak{s}'_{\mathcal{X},-K_{\mathcal{X}},S}(P) + C_\varepsilon. \tag{1}$$

Let $\mathfrak{q} = (q_0, \cdots, q_n)$ be a well-formed set of weights, $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ the weighted projective space over $\mathbb{Q}$ with weights $\mathfrak{q}$, $m = \mathrm{lcm}(q_0, \cdots, q_n)$, and $\mathcal{Z} \subset \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ be a closed subvariety defined by $f_1, \cdots, f_t \in \mathbb{Z}_{\mathfrak{q}}[x_0, \ldots, x_n]$, such that $\mathcal{Z} \cap \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) = \emptyset$, with codimension $r = n - \dim(\mathcal{Z}) \geq 2$ in $\mathcal{X}$. Assume that S is a finite set of primes, $\varepsilon > 0$, and the Vojta's conjecture holds for smooth weighted varieties (see Conj. 4). Then, there exists a nonzero weighted polynomial $g \in \mathbb{Z}_{\mathfrak{q}}[x_0, \ldots, x_n]$ and a constant $\delta = \delta_{\varepsilon,\mathcal{Z}} > 0$, such that every $\tilde{\alpha} = (\alpha_0, \cdots, \alpha_n) \in \mathbb{Z}^{n+1}$ with $\mathrm{wgcd}(\alpha_0, \cdots, \alpha_n) = 1$ satisfies either $g(\tilde{\alpha}) = 0$ or

$$\gcd(f_1(\tilde{\alpha}), \cdots, f_t(\tilde{\alpha})) \leq \max\left\{|\alpha_0|^{\frac{1}{q_0}}, \cdots, |\alpha_n|^{\frac{1}{q_n}}\right\}^{\varepsilon} \cdot (|\alpha_0 \cdots \alpha_n|'_S)^{\frac{1}{m(r-1+\delta)}}, \tag{2}$$

where $|\cdot|'_S$ is the "prime-to-S" part of its origin (cf. Thm. 2).

As a final remark, we mention that the Weighted heights provide a very natural approach of bounding the "size" of points on a weighted variety and the classical projective height is only a special case of weighted heights. It is still unclear whether this new approach of bounding the size of points on weighted projective varieties via weighted heights, instead of the traditional approach of bounding the size of points on projective varieties will lead to any progress toward proving Vojta's conjecture. That remains object of future research.

2 Preliminaries on weighted projective varieties

In this section we provide some background on weighted projective varieties and their heights. For details the reader can consult [13] and its references.

2.1 Weighted projective varieties

Given any integer $n \geq 1$, let $\mathfrak{q} = (q_0, \ldots, q_n)$ be a vector of positive integers. For any field k, $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ denote the affine and projective spaces of dimension n over k, respectively. Consider the polynomial ring $R = k_{\mathfrak{q}}[x_0, \ldots, x_n]$ where x_i has weight q_i for $i = 0, 1, \cdots, n$. We define the **weighted projective space** of dimension n to be the projective scheme $\mathrm{Proj}(R)$, and denote it by $\mathbb{P}^n_{\mathfrak{q},k}$. Notice that if $\mathfrak{q} = (1, \ldots, 1)$, then $\mathbb{P}^n_{\mathfrak{q},k}$ is nothing but the usual projective space $\mathbb{P}^n_k$. We simply write $\mathbb{A}^n$, $\mathbb{P}^n$, and $\mathbb{P}^n_{\mathfrak{q}}$, for the affine, projective and weighted projective spaces of dimension n over $\bar{k}$ an algebraically closed field containing k.

We say that $\mathbb{P}^n_{\mathfrak{q},k}$ is **reduced** if $\gcd(q_0, \cdots, q_n) = 1$. It is called **normalized** or **well-formed** if $\gcd(q_0, \ldots, \hat{q}_i, \ldots, q_n) = 1$, for each $i = 0, \ldots, n$. We notice that any $\mathbb{P}^n_{\mathfrak{q},k}$ is thus isomorphic to a reduced and well-formed one. To see the proof, one can see [13, Prop. 3.3] or [3, Sec. 3C].

A polynomial $f \in R$ is called a **weighted homogeneous of degree** d if every monomial of f has weight d, i.e.

$$f(x_0, \ldots, x_n) = \sum_{i=1}^{t} a_i \prod_{j=0}^{n} x_j^{d_j}, \quad \text{for } a_i \in k \text{ and } t \in \mathbb{N},$$

and for all $0 \leq i \leq n$, we have $\sum_{i=1}^{n} q_i d_j = d$.

By a **weighted variety** we mean an integral, separated subscheme $\mathcal{X}$ of finite type of $\mathbb{P}^n_{\mathfrak{q},k}$, equivalently, if there are weighted homogeneous polynomials $f_1, \cdots, f_t \in R$ such that $\mathcal{X}$ is isomorphic to the k-scheme $\mathrm{Proj}\left(\frac{R}{\langle f_1, \cdots, f_t \rangle}\right)$. For example, a **weighetd hyperplane** in $\mathbb{P}_{\mathfrak{q},k}$ is defined as $\mathrm{Proj}\left(\frac{R}{\langle \ell \rangle}\right)$, where ℓ is a weighted homogeneous polynomial of degree $m = \mathrm{lcm}(q_0, \ldots, q_n)$ of the form

$$\ell(x_0, \ldots, x_n) = a_0 x_0^{m/q_0} + a_1 x_1^{m/q_1} + \cdots + a_n x_n^{m/q_n}, \tag{3}$$

When the positive integer m is an arbitrary multiple of $\mathrm{lcm}(q_0, \ldots, q_n)$, above equation defines a **weighetd diagonal hypersurface** in $\mathbb{P}_{\mathfrak{q},k}$ of degree m, see [6].

2.2 Rational points on weighted varieties in $\mathbb{P}^n_{\mathfrak{q}}$

For $\mathfrak{q} = (q_0, \ldots, q_n)$, we assume that k contains a fixed primitive q_i-th root of unity ξ_{q_i}, which generates a cyclic subgroup of k denoted by μ_{q_i} for every $i = 0, \cdots, n$. Let $G_{\mathfrak{q}} := \mu_{q_0} \times \cdots \times \mu_{q_n}$, which is a finite group of order $|G_{\mathfrak{q}}| = \prod_{i=0}^{n} q_i$. Then, there is an action of $G_{\mathfrak{q}}$ on $\mathbb{P}^n$ given by

$$(\xi_0, \cdots, \xi_n) \bullet [y_0 : \cdots : y_n] = [\xi_0 y_0 : \cdots : \xi_n y_n]\,. \tag{4}$$

The weighted projective space $\mathbb{P}^n_{\mathfrak{q}}$ is isomorphic to the quotient space of this action, i.e., say $\mathbb{P}^n_{\mathfrak{q}} \cong \mathbb{P}^n / G_{\mathfrak{q}}$ as described below. Let $\mathbf{0} = (0, \cdots, 0)$ be the origin of $\mathbb{A}^{n+1}$, and consider the morphism $\pi_0 : \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \longrightarrow \mathbb{A}^{n+1} \setminus \{\mathbf{0}\}$ defined by

$$\pi_0(y_0, \cdots, y_n) = (x_0, \cdots, x_n) := (y_0^{q_0}, \ldots, y_n^{q_n})\,. \tag{5}$$

It induces the following diagram

$$\begin{array}{ccc}
\mathbb{A}^{n+1} \setminus \{\mathbf{0}\} & \xrightarrow{\pi_0} & \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \\
\Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p_{\mathfrak{q}}} \\
\mathbb{P}^n & \xrightarrow{\pi_{\mathfrak{q}}} & \mathbb{P}^n_{\mathfrak{q}} \\
& {\scriptstyle \bar{\pi}_{\mathfrak{q}}} \searrow \quad \nearrow {\scriptstyle \cong} & \\
& \mathbb{P}^n / G_{\mathfrak{q}} &
\end{array} \tag{6}$$

where $p : \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$ is the canonical map $(y_0, \cdots, y_n) \mapsto [y_0 : \cdots : y_n]$, and $p_{\mathfrak{q}} : \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \longrightarrow \mathbb{P}^n_{\mathfrak{q}}$ is given by

$$p_{\mathfrak{q}}(x_0, \cdots, x_n) = p_{\mathfrak{q}}(y_0^{q_0}, \ldots, y_n^{q_n}) = [y_0^{q_0} : \cdots : y_n^{q_n}].$$

Hence, $\bar{\pi}_{\mathfrak{q}} : \mathbb{P}^n \longrightarrow \mathbb{P}^n_{\mathfrak{q}}/G_{\mathfrak{q}}$ is the induced quotient map of the action, which commutes with the map $\pi_{\mathfrak{q}} : \mathbb{P}^n \longrightarrow \mathbb{P}^n_{\mathfrak{q}}$ given by

$$\pi_{\mathfrak{q}}([y_0 : \cdots : y_n]) = [x_0 : \cdots : x_n] := [y_0^{q_0} : \cdots : y_n^{q_n}]. \tag{7}$$

The morphism $\pi_{\mathfrak{q}}$ is surjective, finite, and its fibers are orbits of the action of $G_{\mathfrak{q}}$ on $\mathbb{P}^n$. See [7, Chap. V, Props. 1.3 and 1.8] for more details. Furthermore, $\mathbb{P}^n_{\mathfrak{q}}$ is an irreducible, normal and Cohen-Macaulay variety having only cyclic quotients singularities, see [4].

Conversely, there is a map $\phi_m : \mathbb{P}^n_{\mathfrak{q}} \longrightarrow \mathbb{P}^n$ which is defined by

$$\phi_m([x_0 : \cdots : x_n]) = [y_0 : \cdots : y_n] := [x_0^{m/q_0} : x_1^{m/q_1} : \cdots : x_n^{m/q_n}]. \tag{8}$$

We notice that ϕ_m and $\pi_{\mathfrak{q}}$ are non-canonical maps between $\mathbb{P}^n_{\mathfrak{q}}$ and $\mathbb{P}^n$. If $\mathfrak{q}$ is reduced and all of m/q_i are co-prime, where $m = \operatorname{lcm}(q_0, \cdots, q_i)$, then $\mathbb{P}^n_{\mathfrak{q}}$ is isomorphic to $\mathbb{P}^n$ by via the map ϕ_m and its inverse is $\pi_{\mathfrak{q}}$, see [3, Sec. 3C].

As in the usual projective spaces, a point $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q}}$ is k-rational if and only if $p_{\mathfrak{q}}^{-1}(\mathbf{x})$ is invariant under the Galois group $\operatorname{Gal}(\bar{k}/k)$. We denote the subset of rational points on $\mathbb{P}^n_{\mathfrak{q}}$ by $\mathbb{P}^n_{\mathfrak{q}}(k)$. The orbit of any $x = (x_0, \ldots, x_n) \in \mathbb{A}^{n+1} \setminus \{\mathbf{0}\}$ with image $p_{\mathfrak{q}}(x) = \mathbf{x}$ is the rational curve

$$\mathcal{C}_x^* = p_{\mathfrak{q}}^{-1}(\mathbf{x}) = \left\{(\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n) : \lambda \in \bar{k}^\times\right\} \subset \mathbb{A}^{n+1} \setminus \{\mathbf{0}\}.$$

Moreover, one has $p_{\mathfrak{q}}(x) \in \mathbb{P}^n_{\mathfrak{q}}(k)$ if and only if $\mathcal{C}_x^* \cap \left(\mathbb{A}^{n+1} \setminus \{\mathbf{0}\}\right)(k) \neq \emptyset$. In other words, the map

$$p_{\mathfrak{q}} : \left(\mathbb{A}^{n+1} \setminus \{\mathbf{0}\}\right)(k) \longrightarrow \mathbb{P}^n_{\mathfrak{q}}(k) \tag{9}$$

is surjective and induces a bijection

$$\left(\mathbb{A}^{n+1} \setminus \{\mathbf{0}\}\right)(k)/\sim \ \xrightarrow{\sim} \mathbb{P}^n_{\mathfrak{q}}(k),$$

where $\sim$ denotes an equivalence relation on $\left(\mathbb{A}^{n+1} \setminus \{\mathbf{0}\}\right)(k)$ defined as follows: for two points $x = (x_0, \ldots, x_r)$ and $x' = (x'_0, \ldots, x'_n)$ in $\left(\mathbb{A}^{n+1} \setminus \{\mathbf{0}\}\right)(k)$, we write $x \sim x'$ if there exists some $\lambda \in \bar{k}^*$ such that $(x_0, \ldots, x_r) = (\lambda^{q_0} x'_0, \ldots, \lambda^{q_n} x'_n)$. These can be proved by Hilbert's Theorem 90, as in the usual projective case, for more details one can see [6, Lemma 1.2].

For any weighted subvariety $\mathcal{X} \subseteq \mathbb{P}^n_{\mathfrak{q}}$, the **punctured affine cone** over $\mathcal{X}$ is defined as $\mathcal{C}_{\mathcal{X}}^* = p_{\mathfrak{q}}^{-1}(\mathcal{X})$. The **affine cone** $\mathcal{C}_{\mathcal{X}}$ over $\mathcal{X}$ is the closure of $\mathcal{C}_{\mathcal{X}}^*$ in $\mathbb{A}^{n+1}$. The origin point $\mathbf{0}$ refers to the vertex of $\mathcal{C}_{\mathcal{X}}$.

A point $\mathbf{x} \in \mathcal{X}$ is a k-rational point on $\mathcal{X}$ if $\iota(\mathbf{x}) = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q}}(k)$, where $\iota : \mathcal{X} \to \mathbb{P}^n_{\mathfrak{q}}$ is the natural embedding. We denote by $\mathcal{X}(k)$ the set of all k-rational points on $\mathcal{X}$. Thus, the following diagrams commute

$$\begin{array}{ccc} \mathcal{C}^*_{\mathcal{X}} & \longrightarrow & \mathbb{A}^{n+1}\setminus\{\mathbf{0}\} \\ \downarrow{\scriptstyle p_{\mathfrak{q}}} & & \downarrow{\scriptstyle p_{\mathfrak{q}}} \\ \mathcal{X} & \xrightarrow{\ \iota\ } & \mathbb{P}^n_{\mathfrak{q}} \end{array} \qquad \begin{array}{ccc} \mathcal{C}^*_{\mathcal{X}}(k) & \longrightarrow & \left(\mathbb{A}^{n+1}\setminus\{\mathbf{0}\}\right)(k) \\ \downarrow{\scriptstyle p_{\mathfrak{q}}} & & \downarrow{\scriptstyle p_{\mathfrak{q}}} \\ \mathcal{X}(k) & \xrightarrow{\ \iota\ } & \mathbb{P}^n_{\mathfrak{q}}(k) \end{array} \tag{10}$$

2.3 Weighted heights on $\mathbb{P}^{\boldsymbol{n}}_{\mathfrak{q}}$

Let k be an algebraic number field as in the previous section and $\bar{k}$ be an algebraically closed field containing k. Denote by M_k the set of all places of k, i.e. the equivalent classes of absolute values on k. It is a disjoint union of M_k^0, the set of all non-archimedian places, and M_k^∞, the set of all Archimedean places of k. More precisely, if $\nu \in M_k^0$, then $\nu|_{\mathbb{Q}}$ is the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$. If $\nu \in M_k^\infty$, then $\nu = \nu_\infty$ and $\nu_\infty|_{\mathbb{Q}}$ is the usual absolute value $|\cdot|_\infty$ on $\mathbb{Q}$.

Given any $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(k)$, the **multiplicative weighted height** over k is defined as

$$\mathcal{S}_k(\mathbf{x}) := \prod_{\nu\in M_k} \max\left\{ |x_0|_\nu^{\frac{1}{q_0}}, \ldots, |x_n|_\nu^{\frac{1}{q_n}} \right\} \tag{11}$$

and its **logarithmic weighted height** (over k) as

$$\mathfrak{s}_k(\mathbf{x}) := \log \mathcal{S}_k(\mathbf{x}) = \sum_{\nu\in M_k} \max_{0\le j\le n} \left\{ \frac{1}{q_j} \cdot \log |x_j|_\nu \right\}. \tag{12}$$

In [2, Prop. 1] it is shown that height functions $\mathcal{S}_k(\mathbf{x})$ and hence $\mathfrak{s}_k(\mathbf{x})$ are independent of the choice of coordinates of the point $\mathbf{x}$. Moreover, in [2, Prop. 5-ii], it is proved that for any finite extension $K|k$ we have

$$\mathcal{S}_k(\mathbf{x})^{[K:k]} = \mathcal{S}_K(\mathbf{x}), \text{ and hence } [K:k]\cdot \mathfrak{s}_k(\mathbf{x}) = \mathfrak{s}_K(\mathbf{x}).$$

The **absolute weighted height** on $\mathbb{P}^n_{\mathfrak{q}}(\bar{k})$ is defined as

$$\begin{aligned} \mathcal{S} : \mathbb{P}^n_{\mathfrak{q}}(\bar{k}) &\to [0,\infty], \\ \mathbf{x} &\mapsto \mathcal{S}(\mathbf{x}) := \mathcal{S}_K(\mathbf{x})^{1/[K:k]}, \end{aligned} \tag{13}$$

and the **absolute logarithmic weighted height** on $\mathbb{P}^n_{\mathfrak{q}}(\bar{k})$ is given by

$$\begin{aligned} \mathfrak{s} : \mathbb{P}^n_{\mathfrak{q}}(\bar{k}) &\to [0,\infty], \\ \mathbf{x} &\mapsto \mathfrak{s}(\mathbf{x}) := \frac{1}{[K:k]} \log \mathcal{S}_K(\mathbf{x}), \end{aligned} \tag{14}$$

for which $K \subset \bar{k}$ is a finite extension of k containing $k(\mathbf{x})$, the **field of definition** of $\mathbf{x}$ defined by

$$k(\mathbf{x}) := k\left(\frac{x_0^{1/q_0}}{x_i^{1/q_i}}, \cdots, 1, \cdots, \frac{x_n^{1/q_n}}{x_i^{1/q_i}}\right),$$

for some $x_i \neq 0$. Notice that both of these height functions are independent of the choice of the field K; see [2]. For simplicity, we call $\mathfrak{s}(\mathbf{x})$ the **global weighted height** on $\mathbb{P}^n_{\mathfrak{q}}(\bar{k})$. For all $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(\bar{k})$, we have

$$\mathcal{S}(\mathbf{x}) = H\left(\phi_m(\mathbf{x})\right)^{\frac{1}{m}}, \quad \text{and} \quad \mathfrak{s}(\mathbf{x}) = \frac{1}{m} \cdot h\left(\phi_m(\mathbf{x})\right), \tag{15}$$

where ϕ_m is as in (8), $H(\cdot)$, $h(\cdot)$ are multiplicative and logarithmic heights $\mathbb{P}^n$, and $\mathcal{S}(\cdot)$, $\mathfrak{s}(\cdot)$ are defined as in (13) and (14).

We notice that the above definition of weighted heights on is compatible with those is defined by Ellenberg and et. al., in [5], with respect to a vector bundle on a proper algebraic stack with finite diagonal over a global field.

We end up this section by providing an example of a weighted hypersurface and its rational points of small weighted height illustrating that it would be difficult to obtain this result by embedding this to a projective space counting rational points on that projective space.

Example 1. *Consider* $\mathfrak{q} = (2, 4, 6, 10)$ *and a weighted hypersurface in* $\mathbb{P}^3_{\mathfrak{q},\mathbb{Q}}$ *given by the following equation*

$$\begin{aligned}
&41472wy^5 + 159y^6x^3 - 236196w^2x^5 - 80y^7x + 104976000w^2x^2z - 1728y^5x^2z + 6048y^4xz^2 \\
&- 9331200wy^2z^2 - 2099520000w^2yz + 12x^6y^3z - 54x^5y^2z^2 + 108x^4yz^3 + 1332x^4y^4z \\
&- 8910x^3y^3z^2 + 29376x^2y^2z^3 - 47952xyz^4 - x^7y^4 - 81x^3z^4 - 78x^5y^5 + 384y^6z - 6912y^3z^3 \\
&+ 507384000w^2y^2x - 19245600w^2yx^3 - 592272wy^4x^2 + 77436wy^3x^4 + 4743360wy^3xz \\
&- 870912wy^2x^3z + 3090960wyx^2z^2 - 5832wx^5yz - 125971200000w^3 + 31104z^5 + 972wx^6y^2 \\
&+ 8748wx^4z^2 - 3499200wxz^3 = 0
\end{aligned}$$

This is the locus $\mathcal{L}_2$ *of genus two curves with extra involution where* $(x, y, z, w) = (J_2, J_4, J_6, J_{10})$, *see [14]. The map* $\phi_{60} : \mathbb{P}^3_{\mathfrak{q},\mathbb{Q}} \to \mathbb{P}^3_{\mathbb{Q}}$ *is given by*

$$\mathbf{x} = [x : y : z : w] \to [x^{30} : y^{15} : z^{10} : w^6].$$

There is a one to one correspondence between the rational points $\mathcal{L}_2(\mathbb{Q}) \subset \mathbb{P}_{\mathfrak{q}}(\mathbb{Q})$ *and the rational points of* $\phi_{60}(\mathcal{L}_2) \subset \mathbb{P}^3_{\mathbb{Q}}$. *An inverse of this map is given in [12].*

In [14], it is proved that the above weighted hypersurface has no rational points of weighted height $\mathcal{S} \leq 2$. *To prove this result via the projective varieties we have to check the rational points of* $\phi_{60}(\mathcal{L}_2) \subset \mathbb{P}^3_{\mathbb{Q}}$. *Since,* $H(\phi_m(\mathbf{x})) = \mathcal{S}(\mathbf{x})^m$, *this is equivalent to checking for rational points of projective height* $H(\phi_m(\mathbf{x})) \leq 2^{60}$.

The example above illustrates yet again how efficient computations in the weighted projective spaces are compared to the projective space. □

3 Vojta's conjecture for projective varieties

In this section, we recall the Vojta's famous conjecture on algebraic points of projective varieties, as well as its variants using a correction term involving a multiplier ideal sheaf and its version for log pairs of projective varieties. The interested readers can find more in [15, 16, 17, 19].

3.1 Vojta's conjecture for smooth varieties

Here we give a brief overview of Vojta's conjecture over projective varieties defined over $\mathbb{Q}$. For any finite extension L of $\mathbb{Q}$, we define the **logarithmic discriminant** $d(L)$ by

$$d(L) := \frac{1}{[L:\mathbb{Q}]} \log |\operatorname{Disc}(L)|.$$

Given a variety $\mathcal{X}$ over $\mathbb{Q}$ and a point $\mathbf{x} \in \mathcal{X}$, we define its **logarithmic discriminant** by $d(\mathbf{x}) := d(\mathbb{Q}(\mathbf{x}))$.

Recall that a Cartier divisor D on a smooth projective variety $\mathcal{X}$ is a **normal crossing divisor** if at every point in the support of D there are local coordinates $z_0, z_1, \ldots, z_n$ such that D is given locally by an equation of the form $z_0 z_1 \ldots z_n = 0$. Furthermore, the **canonical divisor** of $\mathcal{X}$ is a divisor $K_{\mathcal{X}}$ such that $\mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}}) = \omega_{\mathcal{X}}$, where $\omega_{\mathcal{X}}$ is the canonical sheaf of regular forms on $\mathcal{X}$. Paul Vojta made a conjecture on algebraic points on projective varieties as follows:

Conjecture 1. *Let $\mathcal{X}$ be a smooth projective variety defined over $\mathbb{Q}$, $K_{\mathcal{X}}$ a canonical divisor, $\mathcal{A}$ an ample divisor, and D a normal crossings divisor on $\mathcal{X}$. Furthermore, let S be a finite subset of places of $\mathbb{Q}$ containing $M_{\mathbb{Q}}^{\infty}$. Then, given any real constant $\varepsilon > 0$ and any positive integer r, there exists a proper Zariski-closed subset Z of $\mathcal{X}$, depending only on $\mathcal{X}, D, \mathcal{A}, \varepsilon$, and r, such that*

$$h_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \lambda_D(\mathbf{x}, \nu) \leq \varepsilon h_{\mathcal{A}}(\mathbf{x}) + d(\mathbf{x}) + O(1),$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{\mathbb{Q}})$ with $[\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq r$.

The case $r = 1$ of the above conjecture is known in the literature as Vojta's conjecture for the rational points of algebraic varieties.

3.2 Vojta's conjecture using a multiplier ideal sheaf

In [17], Vojta restated his conjecture using a correction term involving a multiplier ideal sheaf instead of the normal crossing divisors as follows.

Let $\mathcal{I}$ be a nonzero sheaf of ideals on a projective variety $\mathcal{X}$ and c some positive real constant. Let $f : \mathcal{X}' \to \mathcal{X}$ be a proper birational morphism such that $\mathcal{X}'$ is a smooth variety and $f^*\mathcal{I} = \mathcal{O}_{\mathcal{X}'}(-E)$, for some normal crossing divisor E on $\mathcal{X}'$. Denote by $\mathcal{R}_{\mathcal{X}'/\mathcal{X}}$ the ramification divisor of $\mathcal{X}'$ over $\mathcal{X}$ and define the **multiplier ideal sheaf** $\mathcal{I}_c$ and $\mathcal{I}_c^-$ associated to $\mathcal{I}$ and a rational number c as

$$\mathcal{I}_c := f_* \mathcal{O}_{\mathcal{X}'}(\mathcal{R}_{\mathcal{X}'/\mathcal{X}} - \lfloor cE \rfloor), \text{ and } \mathcal{I}_c^- := \lim_{\varepsilon \to 0^+} \mathcal{I}_{c-\varepsilon},$$

where $\lfloor cE \rfloor$ denotes the round down of the divisor cE. We will denote $\mathcal{I}_1$ and $\mathcal{I}_1^-$ by $\mathcal{I}$ and $\mathcal{I}^-$ respectively.

Conjecture 2. *Let $\mathcal{X}$ be a smooth projective variety defined over $\mathbb{Q}$, $K_{\mathcal{X}}$ a canonical divisor, $\mathcal{A}$ an ample divisor, and $\mathcal{I}$ a nonzero ideal sheaf $\mathcal{X}$. Let S be a finite subset of places of $\mathbb{Q}$ containing $M_{\mathbb{Q}}^{\infty}$. Then, given any real constant $\varepsilon > 0$ and positive integer r, there exists a proper Zariski-closed subset Z of $\mathcal{X}$, depending only on $\mathcal{X}, \mathcal{I}, \mathcal{A}, \varepsilon, r$, such that*

$$h_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \lambda_{\mathcal{I}}(\mathbf{x}, \nu) - \sum_{\nu \in S} \lambda_{\mathcal{I}^-}(\mathbf{x}, \nu) \leq \varepsilon h_{\mathcal{A}}(\mathbf{x}) + d(\mathbf{x}) + O(1).$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{\mathbb{Q}})$ with $[\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq r$.

3.3 Vojta's conjecture for log pairs over $\mathbb{Q}$

Since Conj. 1, and Conj. 2, do not deal with singular varieties in [19] Yasuda formulated a generalization of it in terms of log pairs and variants of multiplier ideals. In order to state his generalization, first we need to recall some terminology. The reader can refer to [9, 10, 11], or [19] for more details.

Let $\mathcal{X}$ be a variety defined over $\mathbb{Q}$. Then, $\mathcal{X}$ is said to be **$\mathbb{Q}$-Gorenstein** if it is Gorenstein in codimension one, satisfies Serre's condition S_2, and a canonical divisor $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. For example, if $\mathcal{X}$ is normal, then the first two conditions are true automatically and hence a canonical divisor exists unequally up to linear equivalence and is Cartier in codimension one.

A **$\mathbb{Q}$-subscheme** of $\mathcal{X}$ is a formal linear combination $\mathcal{Y} = \sum_{i=1}^{m} c_i \cdot \mathcal{Y}_i$ of proper closed subschemes $\mathcal{Y}_i \subset \mathcal{X}$ with all $c_i \in \mathbb{Q}$. The support of such $\mathcal{Y}$ is defined to be the closed subset $\cup_{c_i \neq 0} \mathcal{Y}_i$, and it is called **effective $\mathbb{Q}$-subscheme** if $c_i \geq 0$ for every i. By a **log pair**, we mean a pair $(\mathcal{X}, \mathcal{Y})$ of a $\mathbb{Q}$-Gorenstein variety and an effective $\mathbb{Q}$-subscheme $\mathcal{Y}$ of $\mathcal{X}$. For example, if $\mathcal{X}$ is a normal $\mathbb{Q}$-Gorenstein and D is an effective $\mathbb{Q}$-divisor, then $(\mathcal{X}, D)$ is a log pair.

A **resolution** of $\mathcal{X}$ over $\mathbb{Q}$ is a projective birational morphism $f : \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is a smooth projective variety over $\mathbb{Q}$. By a **log resolution** of a log pair $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{Y} = \sum_{i=1}^{m} c_i \cdot \mathcal{Y}_i$, we mean a resolution $f : \mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}$ such that the set-theoretic inverse image $f^{-1}(\mathcal{Y}_i)$ is a Cartier divisor on $\mathcal{X}'$, and the union of exceptional divisor $\mathrm{Exc}(f)$ of f with all $f^{-1}(\mathcal{Y}_i)_{\mathrm{red}}$ is a simple normal crossing divisor of $\mathcal{X}'$. For a log resolution $f : \mathcal{X}' \to \mathcal{X}$ of a log pair $(\mathcal{X}, \mathcal{Y})$, the **relative canonical divisor** of $\mathcal{X}'$ over $(\mathcal{X}, \mathcal{Y})$ is defined to be the $\mathbb{Q}$-Weil divisor

$$K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} = K_{\mathcal{X}'/\mathcal{X}} - f^*\mathcal{Y},$$

where $K_{\mathcal{X}'/\mathcal{X}}$ is the relative canonical divisor of $\mathcal{X}'$ over $\mathcal{X}$, and $f^*\mathcal{Y}$ is the pull-back of $\mathcal{Y}$ by f over $\mathcal{X}'$.

For a log pair $(\mathcal{X}, \mathcal{Y})$ with a log resolution $f : \mathcal{X}' \to \mathcal{X}$, we define $\mathcal{I}(\mathcal{X}, \mathcal{Y})$ a **variant of multiplier sheaf** as

$$\mathcal{I}(\mathcal{X}, \mathcal{Y}) := f_* \mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil)$$

if $\mathcal{X}$ is a normal variety, otherwise, we let

$$\mathcal{I}(\mathcal{X},\mathcal{Y}) := \bar{f}_*\mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}\rceil),$$

where $\lceil\cdot\rceil$ denotes the round up of the $\mathbb{Q}$-divisors, and $\bar{f}_*\mathcal{O}_{\mathcal{X}'}(E)$ denotes the largest ideal sheaf in $\mathcal{O}_{\mathcal{X}}$ for which its pull-back by f is contained in $\mathcal{O}_{\mathcal{X}'}(E)$ as an $\mathcal{O}_{\mathcal{X}'}$-submodule of (constant) function field sheaf $\mathcal{M}_{\mathcal{X}'}$. Moreover, there exist a constant $\varepsilon_0 > 0$ such that for every rational number $0 < \varepsilon \leq \varepsilon_0$, one has $\mathcal{I}(\mathcal{X},(1-\varepsilon)\mathcal{Y}) = \mathcal{I}(\mathcal{X},(1-\varepsilon_0)\mathcal{Y})$. Based on this fact, we set

$$\mathcal{I}^-(\mathcal{X},\mathcal{Y}) := \mathcal{I}(\mathcal{X},(1-\varepsilon)\mathcal{Y}),\ (0<\varepsilon\ll 1).$$

We also define another ideal sheaf as

$$\mathcal{H}(\mathcal{X},\mathcal{Y}) := \bar{f}_*\mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}\rfloor),$$

where $\bar{f}_*$ is as above. The definitions of $\mathcal{I}(\mathcal{X},\mathcal{Y})$, and hence $\mathcal{I}^-(\mathcal{X},\mathcal{Y})$, as well as $\mathcal{H}(\mathcal{X},\mathcal{Y})$ are independent of the choice of a log resolution by [19, Lem. 3.1] and [19, Prop. 3.4] respectively.

For a $\mathbb{Q}$-Gorenstein projective variety $\mathcal{X}$ with a canonical divisor $K_{\mathcal{X}}$, we can define a global height function $h_{K_{\mathcal{X}}}$ up to addition of a bounded function. For a log pair $(\mathcal{X},\mathcal{Y})$ of a $\mathbb{Q}$-Gorenstein $\mathcal{X}$, we define the logarithmic height function associated to the subscheme $K_{(\mathcal{X},\mathcal{Y})} := K_{\mathcal{X}} + \mathcal{Y}$ as

$$h_{K_{(\mathcal{X},\mathcal{Y})}} = h_{K_{\mathcal{X}}} + h_{\mathcal{Y}}, \tag{16}$$

where $h_{\mathcal{Y}}$ is the logarithmic height function associated to $\mathcal{Y}$ or its ideal sheaf. Next is Yasuda's generalization of Vojta's conjecture for algebraic points.

Conjecture 3. *Let $(\mathcal{X},\mathcal{Y})$ a log pair with projective $\mathcal{X}$, $\mathcal{Y}$ a closed subscheme with ideal sheaf $\mathcal{I} = \mathcal{I}(\mathcal{Y})$, $K_{\mathcal{X}}$ a canonical divisor and $\mathcal{A}$ an ample divisor on $\mathcal{X}$ all defined over $\mathbb{Q}$. Let S be a finite subset of places of $\mathbb{Q}$ containing $M_{\mathbb{Q}}^{\infty}$. Then, given any real constant $\varepsilon > 0$ and positive integer r, there exists a proper Zariski-closed subset Z of $\mathcal{X}$, depending only on $\mathcal{X},\mathcal{I},\mathcal{A},\varepsilon,r$, such that*

$$h_{K_{(\mathcal{X},\mathcal{Y})}} - \sum_{\nu\in S}\lambda_{\mathcal{H}}(\mathbf{x},\nu) - \sum_{\nu\in S}\lambda_{\mathcal{I}^-}(\mathbf{x},\nu) \leq \varepsilon h_{\mathcal{A}}(\mathbf{x}) + d(\mathbf{x}) + O(1).$$

for all $\mathbf{x}\in(\mathcal{X}\setminus Z)(\bar{\mathbb{Q}})$ with $[\mathbb{Q}(\mathbf{x}):\mathbb{Q}]\leq r$, where $\mathcal{H} = \mathcal{H}(\mathcal{X},\mathcal{Y})$ and $\mathcal{I}^- = \mathcal{I}^-(\mathcal{X},\mathcal{Y})$.

One can see that the above conjecture holds for a log pair $(\mathcal{X},\mathcal{Y})$ and a log resolution $f:\mathcal{X}'\to\mathcal{X}$, if Conj. 1, holds for $\mathcal{X}'$ and the reduced simple normal crossing divisor

$$\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}\rceil - \lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}\rfloor,$$

for $0<\varepsilon\ll 1$, where $\lfloor\cdot\rfloor$ denotes the round down of the divisor. Moreover, the Conjecture 1, Conjecture 2, and Conjecture 3 are equivalent; see [19, Prop. 5.4 and Rem. 5.5] for a proof.

4 Vojta's conjecture for weighted projective varieties

In this section, we will state the analogues of Vojta's conjectures given in previous section for weighted projective varieties. In order to do that we review and fix some terminologies on weighted varieties.

4.1 Local and global weighted heights

Let $\mathcal{X}$ be a weighted variety over k, i.e. an integral separated scheme of finite type over $\mathrm{Spec}(k)$. In [13, Prop. 4.3], It is proved that any line bundle $\mathcal{L}$ on $\mathcal{X}$ admits a locally bounded weighted M-metric $\|\cdot\| = (\|\cdot\|_u)$. One can see [13, Sec. 4], for more details on the subject. Given any weighted Cartier divisor $D = \{(U_i, f_i)\}$ on $\mathcal{X}$, we let $\mathcal{L}_D = \mathcal{O}_{\mathcal{X}}(D)$ be the line bundle of regular functions on D, which can be constructed by gluing

$$\mathcal{O}_{\mathcal{X}}(D)|_{U_i} = f_i^{-1}\mathcal{O}_{\mathcal{X}}(U_i)$$

and 1 becomes a canonical invertible meromorphic section of $\mathcal{L}_D$, which is denoted by g_D. Thus, we can equip $\mathcal{L}_D$ with a weighted locally bounded M-metric $\|\cdot\|$, determined by the max-min method in proof of [13, Prop. 4.3], and denote it by $\widehat{D} = (\mathcal{L}_D, \|\cdot\|)$.

For each $\nu \in M_k$, the **local weighted height** $\zeta_{\widehat{D}}(-,\nu)$ **on** $\mathcal{X}$ **with respect to** $\widehat{D}$ on weighted variety $\mathcal{X}$ is defined as

$$\zeta_{\widehat{D}}(\mathbf{x},\nu) = -\log\|g_D(\mathbf{x})\|_v,$$

for $\mathbf{x} \in \mathcal{X}\backslash\,\mathrm{Supp}(D)$, where $v \in M$ such that $\nu = v|_k$. The main properties of local weighted heights are proved in [13, Thm. 4.4]. The **global weighted height** $\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x})$ on $\mathcal{X}$ with respect to $\widehat{\mathcal{L}}$ is defined by

$$\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x}) := \sum_{u\in M_K} \zeta_{\widehat{\mathcal{L}_g}}(\mathbf{x},u),$$

where $\zeta_{\widehat{\mathcal{L}_g}}(\mathbf{x},u) = -\log\|g(\mathbf{x})\|_u$, and its properties are described in [13, Thm. 4.5].

Fix a weighted variety $\mathcal{X}$ in $\mathbb{P}^n_{\mathbf{q},\bar{k}}$ defined over k, i.e., an integral and separated subscheme of finite type. Given any closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ over k, we let $\mathcal{I}_{\mathcal{Y}}$ denotes the corresponding sheaf of ideals generated by $f_1,\cdots,f_r \in k_{\mathbf{q}}[x_0,\cdots,x_n]$ with $\deg(f_j) = d_j$ for $j = 1,\cdots r$. Letting $D_j := \mathrm{div}(f_j)$ for $j = 1,\cdots,r$ and $\nu \in M_k$, we define

$$\zeta_{\mathcal{Y}}(\cdot,\nu) : (\mathcal{X}\backslash\mathcal{Y})(k) \to \mathbb{R},$$

the **local weighted height on** $\mathcal{X}$ **associated to** $\mathcal{Y}$, by

$$\zeta_{\mathcal{Y}}(\mathbf{x},\nu) := \min_{1\le j\le r}\{\zeta_{\widehat{D_j}}(\mathbf{x},\nu)\} = \min_{1\le j\le r}\left\{-\log\frac{|f_j(\mathbf{x})|_\nu}{\max_i |x_i|_\nu^{\frac{d_j}{q_i}}}\right\}. \tag{17}$$

By convention, we define $\zeta_{\mathcal{Y}}(\mathbf{x},\nu) = \infty$ for $\mathbf{x} \in \mathcal{Y}(k)$.

The **global weighted height on $\mathcal{X}$ associated to $\mathcal{Y}$**, can be defined up to a bounded function by summing all local weighted heights. More precisely, given $\mathbf{x} \in \mathcal{X}$, we let K be a finite extension of k containing $k(\mathbf{x})$ and define:

$$\mathfrak{s}_{\mathcal{Y}}(\mathbf{x}) := \sum_{u \in M_K} \zeta_{\mathcal{Y}}(\mathbf{x}, u), \tag{18}$$

which is independent of the choice of the field K.

We notice that for any weighted ideal sheaf $\mathcal{I} \subset k_{\mathfrak{q}}[x_0, \cdots, x_n]$, we may define $\zeta_{\mathcal{I}}$ and $\mathfrak{s}_{\mathcal{I}}$ as $\zeta_{\mathcal{Y}}$ and $\mathfrak{s}_{\mathcal{Y}}$ where $\mathcal{Y} = \mathcal{Z}(\mathcal{I})$ is the weighted variety define by $\mathcal{I}$. The basic properties of weighted local and global heights associated to closed subschemes are respectively described in Propositions 4.6 and 4.7 of [13].

4.2 Weil and Cartier divisors on weighted varieties

Let $\mathcal{X}$ be a weighted variety in $\mathbb{P}^n_{\mathfrak{q},k}$ over the field k. The group of **Weil divisors** on $\mathcal{X}$ is a free Abelian group generated by weighted closed subvarieties of codimension one on $\mathcal{X}$. This group is denoted by $\mathrm{WeDiv}_{\mathfrak{q}}(\mathcal{X})$. The **support** of a Weil divisor $D = \sum_Y n_{\mathcal{Y}} \cdot \mathcal{Y}$ is the union of all $\mathcal{Y}$'s such that $n_{\mathcal{Y}} \neq 0$, which is denoted by $\mathrm{Supp}(D)$. Such a divisor is said to be **effective** if every $n_{\mathcal{Y}} \geq 0$ for all codimension one subvarieties $\mathcal{Y} \subset \mathcal{X}$.

We define $\mathrm{ord}_{\mathcal{Y}} : \mathcal{O}_{\mathcal{X},\mathcal{Y}} \setminus \{0\} \to \mathbb{Z}$ to be

$$\mathrm{ord}_{\mathcal{Y}}(f) = \mathrm{length}_{\mathcal{O}_{\mathcal{X},\mathcal{Y}}} \left(\frac{\mathcal{O}_{\mathcal{X},\mathcal{Y}}}{\langle f \rangle} \right),$$

which is well defined since $\mathcal{O}_{\mathcal{X},\mathcal{Y}}$ is a local ring. Then, one can extend $\mathrm{ord}_{\mathcal{Y}}$ to the fraction field $k_{\mathfrak{q}}(\mathcal{X})^*$ in the usual way. The order function $\mathrm{ord}_{\mathcal{Y}} : k_{\mathfrak{q}}(\mathcal{X})^* \to \mathbb{Z}$ has the following properties:

(i) $\mathrm{ord}_{\mathcal{Y}}(f \cdot g) = \mathrm{ord}_{\mathcal{Y}}(f) + \mathrm{ord}_{\mathcal{Y}}(g)$
(ii) For a fixed $f \in k_{\mathfrak{q}}(\mathcal{X})^*$ there are only finitely many $\mathcal{Y}$ such that $\mathrm{ord}_{\mathcal{Y}} \neq 0$.
(iii) Let $f \in k_{\mathfrak{q}}(\mathcal{X})^*$. Then, $f \in \mathcal{O}_{\mathcal{X},\mathcal{Y}}$ if and only if $\mathrm{ord}_{\mathcal{Y}}(f) \geq 0$. Similarly, $f \in \mathcal{O}^*_{\mathcal{X},\mathcal{Y}}$ if and only if $\mathrm{ord}_{\mathcal{Y}}(f) = 0$.
(iv) If $\mathcal{X}$ is weighted projective variety and $f \in k_{\mathfrak{q}}(\mathcal{X})^*$, then $f \in k^*$ if and only if $\mathrm{ord}_{\mathcal{Y}}(f) \geq 0$ for all $\mathcal{Y}$; if and only if $\mathrm{ord}_{\mathcal{Y}}(f) = 0$ for all $\mathcal{Y}$.

The Weil divisor of any rational function $f \in k_{\mathfrak{q}}(\mathcal{X})^*$ is defined as

$$\mathrm{div}(f) = \sum_{\mathcal{Y} \subset \mathcal{X}} \mathrm{ord}_{\mathcal{Y}}(f) \cdot \mathcal{Y}$$

which is called a **principal divisor** on weighted variety $\mathcal{X}$. Two Weil divisors D and D' on a weighted variety $\mathcal{X}$ are said to be **linearly equivalent** if their difference is a principal divisor. The Weil divisor of zeros and divisor of poles of f, are respectively

defined as:

$$(f)_0 = \sum_{\mathrm{ord}_{\mathcal{Y}}(f)>0} \mathrm{ord}_{\mathcal{Y}}(f)\cdot\mathcal{Y}, \text{ and } (f)_\infty = -\sum_{\mathrm{ord}_{\mathcal{Y}}<0} \mathrm{ord}_{\mathcal{Y}}(f)\cdot\mathcal{Y}$$

The **divisor class group** of $\mathcal{X}$ is the group of divisor classes modulo linear equivalence. This group is denoted by $\mathrm{Cl}_{\mathfrak{q}}(\mathcal{X})$, and $\mathrm{Cl}(\mathbb{P}^n_{\mathfrak{q},k})$ for $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},k}$.

Now, we define a **Cartier divisor** on a weighted variety $\mathcal{X}$ as an equivalence class of collection of pairs $(U_i, f_i)_{i\in I}$ such that the followings hold:

(i) The U_i are affine weighted open sets that cover $\mathcal{X}$.

(ii) The f_i are non zero rational functions, $f_i \in k_{\mathfrak{q}}(U_i)^* = k_{\mathfrak{q}}(\mathcal{X})^*$.

(iii) $\frac{f_i}{f_j} \in \mathcal{O}_{\mathcal{X}}(U_i \cap U_j)^*$, so $\frac{f_i}{f_j}$ has no poles or zeros on $U_i \cap U_j$.

Two Cartier divisors $\{(U_i, f_i)|i \in I\}$ and $\{(V_j, g_j)|j \in J\}$ on a weighted variety $\mathcal{X}$ are said to be equivalent if for all $i \in I$ and $j \in J$ one has

$$\frac{f_i}{g_j} \in \mathcal{O}_{\mathcal{X}}(U_i \cap V_j)^*.$$

The **sum of two Cartier divisors** is defined in as follows:

$$\{(U_i, f_i)|i \in I\} + \{V_j, g_j)|j \in J\} = \{(U_i \cap V_j, f_i\, g_j)|(i, j) \in I \times J\}.$$

The set of Cartier divisors with this operation on a weighted variety $\mathcal{X}$ form a group that we denote it by $\mathrm{CaDiv}_{\mathfrak{q}}(\mathcal{X})$. The **support** of a Cartier divisor is the set of zeros and poles of f_i, which is denoted by $\mathrm{Supp}(D)$. A Cartier divisor is said to be **effective** or **positive** if it can be defined by a collection $\{(U_i, f_i)|i \in I\}$ such that every $f_i \in \mathcal{O}_{\mathcal{X}}(U_i)$. For a given $f \in k_{\mathfrak{q}}(\mathcal{X})^*$, the divisor $\mathrm{div}(f) = \{(\mathcal{X}, f)\}$ is called a **principal Cartier divisor**. Two Cartier divisors are **linearly equivalent** if their difference is a principal divisor. The group of Cartier divisors classes modulo linear equivalence is called **Picard group** of a weighted variety $\mathcal{X}$, which is denoted by $\mathrm{Pic}_{\mathfrak{q}}(\mathcal{X})$, and $\mathrm{Pic}(\mathbb{P}^n_{\mathfrak{q},k})$ when $\mathcal{X} = \mathbb{P}^n_{\mathfrak{q},k}$.

For reduced vector of weights $\mathfrak{q}$, in [1, Sections 5, 6], it is proved that the maps $\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^n_{\mathfrak{q},k})$, given by $1 \mapsto \mathcal{O}_{\mathcal{X}}(1)$, and $\mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^n_{\mathfrak{q},k})$, given by $1 \mapsto \mathcal{O}_{\mathcal{X}}(m)$, with $m = \mathrm{lcm}(q_0, \cdots, q_n)$ induce the isomorphisms $\mathrm{Cl}(\mathbb{P}^n_{\mathfrak{q},k}) \cong \mathbb{Z}$, and $\mathrm{Pic}(\mathbb{P}^n_{\mathfrak{q},k}) \cong \mathbb{Z}$, respectively. Moreover, $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(a)$ is not necessarily an invertible sheaf for any given integer $a \in \mathbb{Z}$. However, the sheaf $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(m)$ with $m = \mathrm{lcm}(q_0, \cdots, q_n)$ is ample and invertible, and for $a, b \in \mathbb{Z}$ we have

$$\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(a) \otimes \mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(m)^{\otimes b} \cong \mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(a + bm).$$

Furthermore, it is proved that $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(m)$ is ample and there is $c \in \mathbb{Z}$ depending on q_i's such that $\mathcal{O}_{\mathbb{P}^n_{\mathfrak{q},k}}(cm)$ is very ample. For more detail on the proofs of above assertions, one can see by [3, Thm. 4B.7 in Sec. 4]. For example, when $\mathfrak{q} = (q_0, q_1, q_2)$ is reduced and well-formed, then $\mathcal{O}_{\mathbb{P}^2_{\mathfrak{q},k}}(m)$ with $m = q_0q_1q_2$ is both ample and very ample.

For $\mathfrak{q} = (1, 6, 10, 15)$ and $m = 30$, the invertible sheaf $\mathcal{O}_{\mathbb{P}^3_{\mathfrak{q},k}}(30)$ in only ample, but $\mathcal{O}_{\mathbb{P}^3_{\mathfrak{q},k}}(30\ell)$ is very ample for all $\ell \geq 2$.

For a subset $J = \{i_0, \cdots, i_r\} \subset \{0, 1, \cdots, n\}$ with $r \leq n$ and a vector of weights $\mathfrak{q} = (q_0, \cdots, q_n)$, we set $\mathfrak{q}_J = (q_{i_0}, \cdots, q_{i_r})$ and $m_J = \mathrm{lcm}\{q_{i_0}, \cdots, q_{i_r}\}$. A Cartier divisor D on a weighted variety $\mathcal{X} \subseteq \mathbb{P}^n_{\mathfrak{q},k}$ is said to be **very ample** if there exist a closed immersion $X \hookrightarrow \mathbb{P}^n_{\mathfrak{q}_J,k}$ for some subset $J = \{i_0, \cdots, i_r\} \subset \{0, 1, \cdots, n\}$ such that $\mathcal{O}_{\mathcal{X}}(D) \cong \mathcal{O}_{\mathbb{P}^r_{\mathfrak{q}_J,k}}(m_J)|_{\mathcal{X}}$. We say that a Cartier divisor D on a weighted variety $\mathcal{X}$ is **ample** if $\ell_J D$ is very ample for some positive large integer ℓ_J. In the rest of paper, we apply the same terminologies to the associated invertible sheaf or line bundle.

4.3 Vojta's conjecture for smooth weighted varieties

We denote by $\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q}})$ the singular locus of $\mathbb{P}^n_{\mathfrak{q}}$. If $[x_0 : \ldots : x_m] \in \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q}})$, then $x_i = 0$ for at least one i. Moreover, if $\gcd(a_j, a_j) = 1$ for every couple (i, j) with $j \neq i$, then $\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q}}) = \{P_0, \ldots, P_m\}$ where P_i are the $m+1$ vertices $[0 : \ldots : 1 : \ldots : 0]$.

A weighted subvariety $\mathcal{X}$ of $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is called **quasi-smooth** of dimension m if its affine cone $\mathcal{C}_{\mathcal{X}}$ is smooth variety of dimension $m+1$ outside its vertex. The singularities of a quasi-smooth variety $\mathcal{X}$ are due to the k^*-action and hence are cyclic quotients singularities. Furthermore, by [3, Cor. 5.9], if $\mathcal{X} \subset \mathbb{P}^n_{\mathfrak{q},k}$ is subvariety such that $\mathcal{X} \cap \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},k}) = \emptyset$, then $\mathcal{X}$ is non-singular if and only if $\mathcal{X}$ is quasi-smooth.

Let $\mathcal{X}$ be a smooth weighted projective variety in $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ defined over $\mathbb{Q}$. Suppose that there is an open subvariety $\mathcal{U} \subset \mathcal{X}$ with complement of codimension at least two which is Gorenstein, i.e., the dualizing sheaf $\omega^0_{\mathcal{U}}$ is invertible. We let $K_{\mathcal{U}}$ be a canonical divisor on $\mathcal{U}$ such that $\omega^0_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}(K_{\mathcal{U}})$, and then define the **canonical sheaf** $K_{\mathcal{X}}$ as the closure of $K_{\mathcal{U}}$ in $\mathcal{X}$.

By a **normal crossing divisor** on weighted $\mathcal{X}$, we mean a Cartier divisor D such that at every point in the support of D such that D is given locally by an equation of the form $x_0^{1/q_0} x_1^{1/q_1} \ldots x_n^{1/q_n} = 0$. Then, we formulate the analogue of Conj. 1, for the smooth weighted projective varieties as follows:

Conjecture 4. *Let $\mathcal{X}$ be a smooth weighted projective variety, $K_{\mathcal{X}}$ a canonical divisor, $\mathcal{A}$ an ample divisor, and D a normal crossings divisor on $\mathcal{X}$, all defined over $\mathbb{Q}$. Let S be a finite subset of places of $\mathbb{Q}$ containing $M^\infty_{\mathbb{Q}}$. Then, given any real constant $\varepsilon > 0$ and any positive integer r, there exists a proper weighted Zariski-closed subset Z of $\mathcal{X}$, depending only on $\mathcal{X}, D, \mathcal{A}, \varepsilon, r$, such that*

$$\mathfrak{s}_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \zeta_D(\mathbf{x}, \nu) \leq \varepsilon \cdot \mathfrak{s}_{\mathcal{A}}(\mathbf{x}) + d(\mathbf{x}) + O(1),$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{\mathbb{Q}})$ with $[\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq r$.

4.4 Vojta's conjecture using a weighted multiplier ideal sheaf

In order to avoid using the weighted normal crossing divisors and replace it by an error term as in [17], let $\mathcal{I}$ be a nonzero weighted ideal sheaf on a weighted projective variety $\mathcal{X}$ and $c \geq 0$ a real constant. For the definition of a weighted ideal, one can see [13]. Let $f : \mathcal{X}' \to \mathcal{X}$ be a proper birational morphism such that $\mathcal{X}'$ is smooth

weighted variety and $f^*\mathcal{I} = \mathcal{O}_{\mathcal{X}'}(-E)$, for a normal crossing divisor E on $\mathcal{X}'$. Denoting by $\mathcal{R}_{\mathcal{X}'/\mathcal{X}}$ the ramification divisor of $\mathcal{X}'$ over $\mathcal{X}$, we define the **weighted multiplier ideal sheaf** $\mathcal{I}_c$ and $\mathcal{I}_c^-$ associated to $\mathcal{I}$ and c as

$$\mathcal{I}_c := f_*\mathcal{O}_{\mathcal{X}'}(\mathcal{R}_{\mathcal{X}'/\mathcal{X}} - \lfloor cE \rfloor), \text{ and } \mathcal{I}_c^- := \lim_{\varepsilon \to 0^+} \mathcal{I}_{c-\varepsilon}. \tag{19}$$

As in the case of projective varieties, we denote $\mathcal{I}_1$ and $\mathcal{I}_1^-$ by $\mathcal{I}$ and $\mathcal{I}^-$, respectively.

An analogue of Conj. 2, for weighted projective varieties follows:

Conjecture 5. *Let $\mathcal{X}$ be a smooth weighted projective variety, $K_{\mathcal{X}}$ a canonical divisor, $\mathcal{A}$ an ample divisor and $\mathcal{I}$ a nonzero weighted ideal sheaf $\mathcal{X}$ all defined over $\mathbb{Q}$. Let S be a finite subset of places of $\mathbb{Q}$ containing $M_{\mathbb{Q}}^{\infty}$. Then, given any real constant $\varepsilon > 0$ and positive integer r, there exists a proper Zariski-closed subset Z of $\mathcal{X}$, depending only on $\mathcal{X}, \mathcal{I}, \mathcal{A}, \varepsilon, r$, such that*

$$\mathfrak{s}_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \zeta_{\mathcal{I}}(\mathbf{x}, \nu) - \sum_{\nu \in S} \zeta_{\mathcal{I}^-}(\mathbf{x}, \nu) \leq \varepsilon\, \mathfrak{s}_{\mathcal{A}}(\mathbf{x}) + d(\mathbf{x}) + O(1),$$

for all $\mathbf{x} \in (\mathcal{X} \setminus Z)(\bar{\mathbb{Q}})$ with $[\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leq r$.

4.5 Vojta's conjecture for weighted log pairs

Next we follow closely the terminology of the log pairs for projective varieties as in section 3.3. A **$\mathbb{Q}$-divisor** on a weighted projective variety $\mathcal{X}$ is a formal finite sum

$$D = \sum c_i D_i.$$

A $\mathbb{Q}$-divisor D on weighted variety $\mathcal{X}$ is said **integral** if all coefficients c_i's are integers. By clearing the denominators of c_i's, we can write $D = cD'$ for some $c \in \mathbb{Q}$ and an integral weighted divisor D'. It is called **effective** if $c_i \geq 0$ and D_i are weighted integral divisors on $\mathcal{X}$. The **support** of D, denoted by $\mathrm{Supp}(D)$, is

$$\mathrm{Supp}(D) = \bigcup \mathrm{Supp}(D_i)$$

as in the case of projective varieties. $D = \sum c_i D_i$ is called **ample** if $c_i \in \mathbb{Q}$, $c_i > 0$ and D_i are all ample Cartier divisors on $\mathcal{X}$. Here, by a **Weil $\mathbb{Q}$-divisor** on a weighted variety $\mathcal{X}$, we mean a $\mathbb{Q}$-linear combination of its codimension one subvarieties, i.e, an element of

$$\mathrm{WeDiv}_{\mathfrak{q}}(\mathcal{X}) \otimes \mathbb{Q}.$$

We use $\lceil D \rceil$ and $\lfloor D \rfloor$ to denote the round up and round down of any Weil $\mathbb{Q}$-divisor $D = \sum_i c_i \mathcal{Y}_i$, that is,

$$\lceil D \rceil = \sum_i \lceil c_i \rceil \mathcal{Y}_i \quad \text{and} \quad \lfloor D \rfloor = \sum_i \lfloor c_i \rfloor \mathcal{Y}_i.$$

A weighted projective variety $\mathcal{X}$ defined over $\mathbb{Q}$ is said to be **$\mathbb{Q}$-Gorenstein** if it is Gorenstein in codimension one, satisfies Serre's condition S_2, and a canonical divisor $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. For example, if $\mathcal{X}$ is normal, then the first two conditions are true and hence a weighted canonical divisor exists unequally up to linear equivalence and is Cartier in codimension one. A **$\mathbb{Q}$-subscheme** of $\mathcal{X}$ is a formal linear combinations

$$\mathcal{Y} = \sum_{i=1}^{m} c_i \cdot \mathcal{Y}_i$$

of proper closed subschemes $\mathcal{Y}_i \subset \mathcal{X}$ with all $c_i \in \mathbb{Q}$. The support of such $\mathcal{Y}$ is defined to be the closed subset $\cup_{c_i \neq 0} \mathcal{Y}_i$, and it is called **effective $\mathbb{Q}$-subscheme** if $c_i \geq 0$ for every i.

By a **resolution** of a weighted variety X over $\mathbb{Q}$, we mean a projective birational morphism $f : \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is a smooth projective weighted variety over $\mathbb{Q}$. The existence of such a weighted resolution is proved by Dolgachev in [4, Subsec. 4.2], by giving a construction of resolution for the sheaf of regular differentials on $\mathcal{X}$. One can see [18], for a Rees algebra approach on weighted resolution.

A **weighted log pair** is called a pair $(\mathcal{X}, \mathcal{Y})$ of a $\mathbb{Q}$-Gorenstein weighted variety $\mathcal{X}$ and an effective weighted $\mathbb{Q}$-subscheme $\mathcal{Y}$ of $\mathcal{X}$. A **weighted log resolution** of a weighted log pair $(\mathcal{X}, \mathcal{Y})$ with

$$\mathcal{Y} = \sum_{i=1}^{m} c_i \cdot \mathcal{Y}_i,$$

is a projective birational morphism $f : \mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}$ such that $\mathcal{X}'$ is a weighted smooth variety defined over $\mathbb{Q}$, the set-theoretic inverse image $f^{-1}(\mathcal{Y}_i)$ is a weighted Cartier divisor on $\mathcal{X}'$, and the union of $\mathrm{Exc}(f)$ of the exceptional divisor of f with all $f^{-1}(\mathcal{Y}_i)_{\mathrm{red}}$ is a simple weighted normal crossing divisor of $\mathcal{X}'$. The existence of a the weighted log resolution of $(\mathcal{X}, \mathcal{Y})$ is a consequence of Hironoka's theorem [10, Thm. 4.1.3].

For a weighted log resolution $f : \mathcal{X}' \to \mathcal{X}$ as a weighted log pair $(\mathcal{X}, \mathcal{Y})$, the **relative weighted canonical divisor** of $\mathcal{X}'$ over $(\mathcal{X}, \mathcal{Y})$ is defined to be the weighted $\mathbb{Q}$-Weil divisor

$$K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} = K_{\mathcal{X}'/\mathcal{X}} - f^*\mathcal{Y},$$

where $f^*\mathcal{Y}$ is the pull-back of $\mathcal{Y}$ by f over $\mathcal{X}'$ and $K_{\mathcal{X}'/\mathcal{X}}$ is the relative canonical divisor of $\mathcal{X}'$ over $\mathcal{X}$, i.e.,

$$\mathcal{O}(K_{\mathcal{X}'}) \cong \mathcal{O}(K_{\mathcal{X}'/\mathcal{X}}) \otimes f^*\mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}}),$$

which is a $\mathbb{Q}$-Weil divisor on $\mathcal{X}'$. Given a weighted log pair $(\mathcal{X}, \mathcal{Y})$ and a weighted log resolution $f : \mathcal{X}' \to \mathcal{X}$, we write

$$K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} = \sum_{\mathbb{Z}} a_{\mathcal{Z}} \cdot \mathcal{Z},$$

where $\mathcal{Z}$ runs over all prime divisors of $\mathcal{X}'$. The weighted log pair $(\mathcal{X}, \mathcal{Y})$ is called **strongly canonical** (resp. **Kawamata log terminal,** and **log canonical**) if $a_{\mathcal{Z}} \geq 0$

(resp. $a_{\mathcal{Z}} > 0$, and $a_{\mathcal{Z}} \geq -1$) for every $\mathcal{Z}$. These properties are independent of the resolution and are also local.

Define the **weighted non-strongly canonical locus (wnsc)** of the pair $(\mathcal{X}, \mathcal{Y})$ to be the smallest weighted closed subset $\mathcal{W} \subset \mathcal{X}$ such that the weighted pair $(\mathcal{X}\backslash\mathcal{W}, \mathcal{Y}|_{\mathcal{X}\backslash\mathcal{W}})$ is strongly canonical. The **weighted non-Kawamata log terminal locus (wnklt)** of the pair $(\mathcal{X}, \mathcal{Y})$ to be the smallest weighted closed subset $\mathcal{W} \subset \mathcal{X}$ such that the weighted pair is Kawamata log terminal. Similarly the **weighted non-log canonical (wnlc)**) of the pair $(\mathcal{X}, \mathcal{Y})$ to be the smallest weighted closed subset $\mathcal{W} \subset \mathcal{X}$ such that the weighted pair is log canonical. We denote them respectively as $\mathrm{wnsc}(\mathcal{X}, \mathcal{Y})$, $\mathrm{wnklt}(\mathcal{X}, \mathcal{Y})$, $\mathrm{wnlc}(\mathcal{X}, \mathcal{Y})$. One may check that

$$\mathrm{wnsc}(\mathcal{X}, \mathcal{Y}) \subset \mathrm{wnklt}(\mathcal{X}, \mathcal{Y}) \subset \mathrm{wnklt}(\mathcal{X}, \mathcal{Y}).$$

For a weighted log pair $(\mathcal{X}, \mathcal{Y})$ with a log resolution $f : \mathcal{X}' \to \mathcal{X}$, we define $\mathcal{I}(\mathcal{X}, \mathcal{Y})$ a **variant of multiplier sheaf** as

$$\mathcal{I}(\mathcal{X}, \mathcal{Y}) := f_*\mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil)$$

if $\mathcal{X}$ is a weighted normal variety; see [11, 9.3.56] for the definition of multiplier ideal sheaf in usual case. Otherwise, we set

$$\mathcal{I}(\mathcal{X}, \mathcal{Y}) := \bar{f}_*\mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rceil),$$

where $\bar{f}_*\mathcal{O}_{\mathcal{X}'}(E)$ denotes the largest ideal sheaf in $\mathcal{O}_{\mathcal{X}}$ for which its pull-back by f is contained in $\mathcal{O}_{\mathcal{X}'}(E)$ as an $\mathcal{O}_{\mathcal{X}'}$-submodule of (constant) function field sheaf $\mathcal{M}_{\mathcal{X}'}$. Moreover, there exist a constant $\varepsilon_0 > 0$ such that for every rational number $0 < \varepsilon \leq \varepsilon_0$, one has $\mathcal{I}(\mathcal{X}, (1-\varepsilon)\mathcal{Y}) = \mathcal{I}(\mathcal{X}, (1-\varepsilon_0)\mathcal{Y})$.

Let

$$\begin{aligned} \mathcal{I}^-(\mathcal{X}, \mathcal{Y}) &:= \mathcal{I}(\mathcal{X}, (1-\varepsilon)\mathcal{Y}), \ (0 < \varepsilon \ll 1) \\ \mathcal{H}(\mathcal{X}, \mathcal{Y}) &:= \bar{f}_*\mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor), \end{aligned} \tag{20}$$

where $\bar{f}_*$ is as above. We note that the definition of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ does not depend on the notion of "weighted simple normal crossing".

Lemma 1. *The definitions of $\mathcal{I}(\mathcal{X}, \mathcal{Y})$, $\mathcal{I}^-(\mathcal{X}, \mathcal{Y})$, and $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ are independent of the choice of a weighted log resolution.*

Proof. By adopting [19, Lem. 3.1] and [19, Prop. 3.4] respectively to the case of weighted projective schemes, one get the result for $\mathcal{I}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{I}^-(\mathcal{X}, \mathcal{Y})$. An argument similar to the proof of the Proposition 3.4 in [19] shows the assertion for $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. □

Proposition 1. *Let $(\mathcal{X}, \mathcal{Y})$ be a weighted log pair. Then, the following are true:*

(i) $\mathrm{wnlc}(\mathcal{X}, \mathcal{Y}) \subset \mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{I}^-(\mathcal{X}, \mathcal{Y})) \subset \mathrm{wnklt}(\mathcal{X}, \mathcal{Y})$;

(ii) *If $(\mathcal{X}\backslash\,\mathrm{Supp}(\mathcal{Y}), 0)$ is a weighted Kawamata log terminal, then*

$$\mathrm{wnlc}(\mathcal{X}, \mathcal{Y}) = \mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{I}^-(\mathcal{X}, \mathcal{Y}));$$

(iii) $\mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{H}(\mathcal{X},\mathcal{Y})) = \mathrm{wnsc}(\mathcal{X},\mathcal{Y})$.

(iv) *If $(\mathcal{X},\mathcal{Y})$ is weighted log canonical, then $\mathcal{O}_{\mathcal{X}}/\mathcal{H}(\mathcal{X},\mathcal{Y})$ is reduced, i.e., as a reduced closed subscheme, $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is the ideal sheaf of* $\mathrm{wnsc}(\mathcal{X},\mathcal{Y})$.

Proof. Let $f : \mathcal{X}' \to \mathcal{X}$ be a weighted log resolution of $(\mathcal{X},\mathcal{Y})$, and denote by $\mathrm{mult}_{\mathcal{Z}}(E)$ the multiplicity of any divisor E on $\mathcal{X}'$.

(i) Given any prime divisor $\mathcal{Z}$ of $\mathcal{X}'$ and real constant $0 < \varepsilon \ll 1$, we have

$$\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) < -1,$$

which implies that

$$\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}) < 0,$$

and hence $\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) \leq -1$. This proves part (i).

(ii) It is enough to show that if the pair $(\mathcal{X}\setminus \mathrm{Supp}(\mathcal{Y}), 0)$ is a weighted Kawamata log terminal, then

$$\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) \geq -1$$

and so

$$\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}) \geq 0.$$

If $\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) > -1$, then the result is trivial. If $\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) = -1$, then $\mathcal{Z}$ is contained in $\mathrm{Supp}(f^*\mathcal{Y})$ by assumption on $(\mathcal{X}\setminus \mathrm{Supp}(\mathcal{Y}), 0)$. Thus,

$$\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}) > -1 \Rightarrow \mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^*\mathcal{Y}) \geq 0.$$

(iii) If $(\mathcal{X},\mathcal{Y})$ is a weighted strongly canonical, then $\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor \geq 0$. By definition of $\bar{f}$, we have

$$\bar{f}_*\mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor) = \mathcal{O}_{\mathcal{X}}.$$

This shows that

$$\mathrm{wnsc}(\mathcal{X},\mathcal{Y}) \subset \mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{H}(\mathcal{X},\mathcal{Y})).$$

If $(\mathcal{X},\mathcal{Y})$ is not a weighted strongly canonical around $x \in \mathcal{X}$, then there is a prime divisor $\mathcal{Z}$ on $\mathcal{X}'$ such that $x \in f(\mathcal{Z})$ and $\mathrm{mult}_{\mathcal{Z}}(K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}) < 0$. Thus, we have

$$\mathcal{O}_{\mathcal{X}'} \not\subset \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor).$$

Replacing $\mathcal{X}$ with any open neighborhood of x does not change the last result. Therefore,

$$\mathrm{Supp}(\mathcal{O}_{\mathcal{X}}/\mathcal{H}(\mathcal{X},\mathcal{Y})) \subset \mathrm{wnsc}(\mathcal{X},\mathcal{Y}).$$

(iv) By part (iii), we have $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is a subset of the ideal sheaf of $\mathrm{wnsc}(\mathcal{X},\mathcal{Y})$ denoted by $\mathcal{N}$. Now, let $\mathcal{U} \subset \mathcal{X}$ be an open set and $g \in \mathcal{N}(\mathcal{U})$. Then f^*g vanishes along the closed set $f^*(\mathrm{wnsc}(\mathcal{X},\mathcal{Y}))$ containing every prime divisor $\mathcal{Z}$ on $\mathcal{X}'$ having negative coefficient in $\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor$, which is equal to -1 since $(\mathcal{X},\mathcal{Y})$ is weighted log canonical. Therefore,

$$f^*g \in \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor)(f^{-1}\mathcal{U}),$$

which implies $g \in \mathcal{H}(\mathcal{U})$ and hence $\mathcal{H} \subset \mathcal{N}$. □

Given a Weil $\mathbb{Q}$-divisor D on a weighted projective variety $\mathcal{X}$, such that nD is a Weil divisor on $\mathcal{X}$, we define weighted height function

$$\mathfrak{s}_D := \frac{1}{n}\mathfrak{s}_{nD}\,.$$

For a weighted $\mathbb{Q}$-Gorenstein projective variety $\mathcal{X}$ with a canonical divisor $K_{\mathcal{X}}$, we can define a weighted global height function $\mathfrak{s}_{K_{\mathcal{X}}}$ up to addition of a bounded function. Given a weighted log pair $(\mathcal{X},\mathcal{Y})$ of a weighted $\mathbb{Q}$-Gorenstein $\mathcal{X}$, we define the weighted height function associated to the subscheme $K_{(\mathcal{X},\mathcal{Y})} = K_{\mathcal{X}} + \mathcal{Y}$ as

$$\mathfrak{s}_{K_{(\mathcal{X},\mathcal{Y})}} = \mathfrak{s}_{K_{\mathcal{X}}} + \mathfrak{s}_{\mathcal{Y}}, \tag{21}$$

where $\mathfrak{s}_{\mathcal{Y}}$ is the weighted height function associated to the subscheme $\mathcal{Y}$ or its ideal sheaf. For more detail on the local and global weighted height functions, we refer the reader to [13].

Next we are ready to state Vojta's conjecture for weighted log pairs.

Conjecture 6. *Let $\mathcal{X}$ be a weighted projective scheme, $\mathcal{Y}$ a closed weighted subscheme with ideal sheaf $\mathcal{I} = \mathcal{I}(\mathcal{Y})$, $K_{\mathcal{X}}$ a canonical divisor, and $\mathcal{A}$ an ample divisor on $\mathcal{X}$, all defined over $\mathbb{Q}$.*

Let $(\mathcal{X},\mathcal{Y})$ be a weighted log pair and S be a finite subset of places of $\mathbb{Q}$ containing $M_{\mathbb{Q}}^{\infty}$. Then, given any real constant $\varepsilon > 0$ and a positive integer r, there exists a proper weighted Zariski-closed subset Z of $\mathcal{X}$, depending only on $\mathcal{X},\mathcal{I},\mathcal{A},\varepsilon,r$, such that

$$\mathfrak{s}_{K_{(\mathcal{X},\mathcal{Y})}(\mathbf{x})} - \sum_{\nu\notin S}\zeta_{\mathcal{H}}(\mathbf{x},\nu) - \sum_{\nu\in S}\zeta_{\mathcal{I}^-}(\mathbf{x},\nu) \le \varepsilon\,\mathfrak{s}_{\mathcal{A}}(\mathbf{x}) + d(\mathbf{x}) + O(1).$$

for all $\mathbf{x}\in(\mathcal{X}\setminus Z)(\bar{\mathbb{Q}})$ with $[\mathbb{Q}(\mathbf{x}):\mathbb{Q}]\le r$, where

$$\mathcal{H} = \mathcal{H}(\mathcal{X},\mathcal{Y}) \quad \text{and} \quad \mathcal{I}^- = \mathcal{I}^-(\mathcal{X},\mathcal{Y}).$$

We note that the terms

$$\sum_{\nu\in S}\zeta_{\mathcal{H}}(\mathbf{x},\nu) \quad \text{and} \quad \sum_{\nu\in S}\zeta_{\mathcal{I}^-}(\mathbf{x},\nu)$$

can be thought of as the contribution of $\mathrm{wnsc}(\mathcal{X},\mathcal{Y})$ and $\mathrm{wnklt}(\mathcal{X},\mathcal{Y})$, or $\mathrm{wnlc}(\mathcal{X},\mathcal{Y})$ if $(\mathcal{X}\backslash(Supp)(\mathcal{Y}),0)$ is Kawamata log terminal.

Since a pair $(\mathcal{X},D)$ with $\mathcal{X}$ a smooth weighted variety and D a reduced simple weighted normal crossing divisor on $\mathcal{X}$ is a weighted log canonical, by parts (i), (iii) and (iv) of Prop. 1, one can conclude that

$$\sum_{\nu\in S}\zeta_{\mathcal{I}^-}(\mathbf{x},\nu) = 0$$

and hence the right hand side of the inequality of Conj. 6 is equal to

$$\mathfrak{s}_{K_{(\mathcal{X},D)}}(\mathbf{x}) - \sum_{\nu \notin S} \zeta_{\mathcal{H}}(\mathbf{x}, \nu) = \mathfrak{s}_{K_{\mathcal{X}}}(\mathbf{x}) + \sum_{\nu \in S} \zeta_{D}(\mathbf{x}, \nu).$$

Thus Conj. 6 is the same as Conj. 5 and Conj. 4, in this case.

In contrast, given a weighted log pair $(\mathcal{X}, \mathcal{Y})$ and a log resolution

$$f : \mathcal{X}' \to \mathcal{X},$$

if we suppose that Conj. 4, holds for $\mathcal{X}'$ and the reduced simple normal crossing divisor

$$\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^{*}\mathcal{Y} \rceil - \lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor,$$

for $0 < \varepsilon \ll 1$, then Conj. 5 and Conj. 6 holds for $(\mathcal{X}, \mathcal{Y})$.

Indeed, the argument is similar to those given in [17, Prop. 4.3] and [19, Prop. 5.4] as follows. By definition, we have

$$f^{-1}\mathcal{H} \subset \mathcal{O}_{\mathcal{X}'}(\lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor)$$

and

$$f^{-1}\mathcal{I}^{-} \subset \mathcal{O}_{\mathcal{X}'}(\lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^{*}\mathcal{Y} \rceil)$$

for $0 < \varepsilon \ll 1$. Using these and the properties of weighted height functions, we get

$$\begin{aligned} &\zeta_{\mathcal{H}} \circ f \geq \zeta_{-D_1}, \text{ with } D_1 = \lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor, \\ &\zeta_{\mathcal{I}^{-}} \circ f \geq \zeta_{-D_2} \text{ with } D_2 = \lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^{*}\mathcal{Y} \rceil. \end{aligned}$$

Then, using the above inequalities, we have

$$\begin{aligned} \left(\mathfrak{s}_{K_{(\mathcal{X},\mathcal{Y})}} - \sum_{\nu \notin S} \zeta_{\mathcal{H}}(\cdot, \nu) - \sum_{\nu \in S} \zeta_{\mathcal{I}^{-}}(\cdot, \nu) \right) \circ f &\leq \mathfrak{s}_{K_{\mathcal{X}'}} - \mathfrak{s}_{K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})}} - \sum_{\nu \notin S} \zeta_{-D_1} - \sum_{\nu \in S} \zeta_{-D_2} \\ &\leq \mathfrak{s}_{K_{\mathcal{X}'}} + \mathfrak{s}_{-D_1} - \sum_{\nu \notin S} \zeta_{-D_1} - \sum_{\nu \in S} \zeta_{-D_2} \\ &\leq \mathfrak{s}_{K_{\mathcal{X}'}} + \sum_{\nu \notin S} \zeta_{D_2 - D_1}, \end{aligned}$$

where

$$D_2 - D_1 = \lceil K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} + \varepsilon f^{*}\mathcal{Y} \rceil - \lfloor K_{\mathcal{X}'/(\mathcal{X},\mathcal{Y})} \rfloor.$$

Recall that the pullback $f^{*}D$ of an ample divisor D is ample. Therefore, the above argument leads to following result.

Corollary 1. *The Conjecture 4, Conjecture 5, and Conjecture 6 are equivalent.*

5 Weighted blow-ups and generalized weighted gcds

In this section, we define the generalized weighted greatest common divisor generalizing the usual one. Then, we describe it as a weighted height for weighted blow-ups using the theory introduced in [13].

For any two integers $\alpha, \beta \in \mathbb{Z}$ the **greatest common divisor** is defined as

$$\gcd(\alpha, \beta) := \prod_{p \text{ prime}} p^{\min\{\nu_p(\alpha),\, \nu_p(\beta)\}},$$

where ν_p is the place associated to the prime number p. The **logarithmic greatest common divisor** is

$$\log \gcd(\alpha, \beta) := \sum_{p \text{ prime}} \min\{\nu_p(\alpha), \nu_p(\beta)\} \log p.$$

For each place $\nu \in M_{\mathbb{Q}}$, we define

$$\begin{aligned} \nu^+ : \mathbb{Q} &\longrightarrow [0, \infty], \\ \alpha &\mapsto \max\{\nu(\alpha), 0\}, \end{aligned} \tag{22}$$

which ν^+ can be viewed as a height function on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ with respect the divisor (0), where we set $\nu^+(\infty) = 0$. The **generalized logarithmic greatest common divisor** of $\alpha, \beta \in \mathbb{Q}$ is defined as

$$\operatorname{hgcd}(\alpha, \beta) := \sum_{\nu \in M_{\mathbb{Q}}} \min\{\nu^+(\alpha), \nu^+(\beta)\}.$$

Then, given $(\alpha, \beta) \neq (0, 0)$, one may consider the following function

$$\begin{aligned} G_{\nu_p} : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) &\to [0, \infty], \\ (\alpha, \beta) &\mapsto \min\{\nu^+(\alpha), \nu^+(\beta)\}, \end{aligned} \tag{23}$$

as a **local height function** and the generalized logarithmic greatest common divisor, being their sum all together,

$$\operatorname{hgcd}(\alpha, \beta) = \sum_{\nu_p \in M_{\mathbb{Q}}} G_{\nu_p}. \tag{24}$$

as a **global height function** on $\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$.

Here is a theoretical interpretation of the function G_ν in terms of blow-ups. More precisely, for $\mathcal{X} = (\mathbb{P}^1_{\mathbb{Q}})^2$ let $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up of the point $(0, 0)$ and $E = \pi^{-1}(0, 0)$ be the exceptional divisor of the blow-up. Then, for all $(\alpha, \beta) \in \mathcal{X}(\mathbb{Q}) \backslash (0, 0)$ and $\nu \in M_{\mathbb{Q}}$, one has

$$\lambda_{\mathcal{X}, E}(\pi^{-1}(\alpha, \beta), \nu) = \min\{\nu^+(\alpha), \nu^+(\beta)\},$$

and adding all together for $\nu \in M_{\mathbb{Q}}$ leads to

$$\operatorname{hgcd}(\alpha, \beta) = h_{\tilde{\mathcal{X}},E}(\pi^{-1}(\alpha, \beta)).$$

5.1 Generalized weighted greatest common divisors as heights for blow-ups

The weighted greatest common divisor for any tuple of integers $(x_0, \cdots, x_n) \in \mathbb{Z}^{n+1}$ was defined in [2], which we are going to recall in below.

Let $\tilde{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ with $r = \gcd(x_0, \ldots, x_n)$ and

$$r = u \cdot \prod_{j=1}^{s} p_j,$$

where $u \in \{-1, 1\}$ and $p_1, \cdots, p_s$ are prime numbers. The **weighted greatest common divisor** of $\tilde{x} \in \mathbb{Z}^{n+1}$ is defined as

$$\operatorname{wgcd}(\tilde{x}) := \prod_{\substack{p \in \{p_1, \cdots, p_s\} \\ p^{q_i} | x_i}} p = \prod_{\nu_p \in M_{\mathbb{Q}}} p^{\min\left\{\left\lfloor \frac{\nu_p(x_0)}{q_0} \right\rfloor, \ldots, \left\lfloor \frac{\nu_p(x_n)}{q_n} \right\rfloor\right\}}, \tag{25}$$

where the last equality comes from [2, Lem. 4]. Here, the symbol $\lfloor \cdot \rfloor$ denotes the integer part function. A tuple $\tilde{x} \in \mathbb{Q}^{n+1}$ is said to be **normalized** if $\operatorname{wgcd}(\tilde{x}) = 1$. In [2, Lem. 7 and Cor. 1], it is proved that any point $\mathbf{x}$ in a well-formed space $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ has a unique normalization $\mathbf{y} = \frac{1}{\operatorname{wgcd}(\tilde{x})} \star \mathbf{x}$. For the rest of paper, a **normalized point** $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ means a point $\mathbf{x} = [x_0 : \ldots : x_n]$ with integer coordinates $x_i \in \mathbb{Z}$ such that $\operatorname{wgcd}(x_0, \ldots, x_n) = 1$.

Given $\mathfrak{q} = (q_0, \cdots, q_n)$, we let $\mathfrak{q}_i = (1, q_i)$ for each $i = 0, 1, \cdots n$. The canonical inclusion

$$\mathbb{Q}_{\mathfrak{q}_i}(x_i) \hookrightarrow \mathbb{Q}_{\mathfrak{q}}(x_0, \cdots, x_n)$$

implies the rational map $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \to \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}}$ given by

$$\mathbf{x} = [x_0 : \cdots : x_n] \mapsto [1 : x_i],$$

which is defined precisely in the complement of $\{x_i = 0\}$ in $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$. Considering all of these maps, we have the rational map

$$\begin{aligned} \varphi_{n,\mathfrak{q}} : \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} &\longrightarrow \prod_{i=0}^{n} \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}}, \\ \mathbf{x} = [x_0 : \cdots : x_n] &\mapsto \varphi_{n,\mathfrak{q}}(\mathbf{x}) := ([1 : x_0], [1 : x_1], \cdots, [1 : x_n]), \end{aligned} \tag{26}$$

which is defined on the open set $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \setminus \cup_{i=0}^n \{x_i = 0\}$. For each $p \in \mathbb{Z}$, we define the function F_{ν_p} as follows,

$$\begin{aligned} F_{\nu_p} : \prod_{i=1}^n \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}} &\longrightarrow \mathbb{N}, \\ ([1:x_0],[1:x_1],\cdots,[1:x_n]) &\mapsto p^{\min\left\{\left\lfloor \frac{\nu_p^+(x_0)}{q_0}\right\rfloor,\ldots,\left\lfloor \frac{\nu_p^+(x_n)}{q_n}\right\rfloor\right\}}. \end{aligned}$$

The **generalized weighted greatest common divisor** of a given tuple $\bar{x} = (x_0,\ldots,x_n) \in \mathbb{Q}^{n+1}$ is defined as

$$\mathrm{h}_{\mathrm{wgcd}}(\bar{x}) := \prod_{\nu_p \in M_{\mathbb{Q}}} p^{\min\left\{\left\lfloor \frac{\nu_p^+(x_0)}{q_0}\right\rfloor,\ldots,\left\lfloor \frac{\nu_p^+(x_n)}{q_n}\right\rfloor\right\}}, \tag{27}$$

We define the **logarithmic weighted greatest common divisor** of any tuple of integers $\tilde{x} = (x_0,\cdots,x_n) \in \mathbb{Z}^{n+1}$ as the sum

$$\log \mathrm{wgcd}(\tilde{x}) := \sum_{\nu \in M^0_{\mathbb{Q}}} \min\left\{\left\lfloor \frac{\nu(x_0)}{q_0}\right\rfloor,\ldots,\left\lfloor \frac{\nu(x_n)}{q_n}\right\rfloor\right\}, \tag{28}$$

and the **generalized logarithmic weighted greatest common divisor** of any tuple $\bar{x} = (x_0,\cdots,x_n) \in \mathbb{Q}^{n+1}$ is defined to be

$$\log \mathrm{h}_{\mathrm{wgcd}}(\bar{x}) := \sum_{\nu \in M_{\mathbb{Q}}} \min\left\{\left\lfloor \frac{\nu^+(x_0)}{q_0}\right\rfloor,\ldots,\left\lfloor \frac{\nu^+(x_n)}{q_n}\right\rfloor\right\}. \tag{29}$$

Let us consider the following positive real-valued function on $\mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$,

$$\begin{aligned} T_\nu : \prod_{i=0}^n \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}} &\to [0,\infty] \\ ([1:x_0],[1:x_n],\ldots,[1:x_n]) &\to \min\left\{\left\lfloor \frac{\nu^+(x_0)}{q_0}\right\rfloor,\ldots,\left\lfloor \frac{\nu^+(x_n)}{q_n}\right\rfloor\right\}. \end{aligned} \tag{30}$$

For any rational point $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$, define its **generalized logarithmic weighted greatest common divisor** as

$$\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = \sum_{\nu \in M_{\mathbb{Q}}} T_\nu(\varphi_{n,\mathfrak{q}}(\mathbf{x})) = \sum_{\nu_p \in M_{\mathbb{Q}}} \min\left\{\left\lfloor \frac{\nu_p^+(x_0)}{q_0}\right\rfloor,\ldots,\left\lfloor \frac{\nu_p^+(x_n)}{q_n}\right\rfloor\right\}, \tag{31}$$

where $\varphi_{n,\mathfrak{q}}$ is defined by (26).

Notice that all points $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$ with $\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = 0$ belong to the singular locus $\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}})$ as shown next.

Proposition 2. *Let $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ be a well-formed weighted projective space with $\mathfrak{q} = (q_0, \cdots, q_n)$ and $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$. If $\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = 0$ then $\mathbf{x} \in \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}})$.*

Proof. Let $m = \mathrm{lcm}(q_0, \cdots, q_n)$, and define $J(\mathbf{x}) = \{j : x_j(\mathbf{x}) \neq 0\}$ for any point $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q}}(\mathbb{Q})$. Given any prime divisor $p \mid m$, we define

$$S_{\mathfrak{q}}(p) = \left\{\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} : \ p \mid q_i \text{ for all } i \in J(\mathbf{x})\right\}.$$

Then $\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) = \bigcup_{p|m} S_{\mathfrak{q}}(p)$ and for any prime p we have

$$\begin{aligned}
\mathbf{x} = [x_0 : \cdots : x_n] \in S_{\mathfrak{q}}(p) &\Rightarrow p \mid q_i, \text{ for all } i \in J(\mathbf{x}) \\
&\Rightarrow \nu_p^+(x_i) < q_i, \text{ for all } i \in J(\mathbf{x}) \\
&\Rightarrow \left\lfloor \frac{\nu_p^+(x_i)}{q_i} \right\rfloor = 0, \text{ for all } i \in J(\mathbf{x}) \\
&\Rightarrow \min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \ldots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\} = 0.
\end{aligned}$$

If we assume $\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) = 0$, then

$$\sum_{\nu_p \in M_{\mathbb{Q}}} \min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \ldots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\} = 0. \tag{32}$$

Thus, for all $\nu_p \in M_{\mathbb{Q}}$ with $p \in \mathbb{Z}$, we have

$$\min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \ldots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\} = 0. \tag{33}$$

This implies that $\mathbf{x} \in S_{\mathfrak{q}}(p)$ for any prime $p \mid m$ and hence $\mathbf{x} \in \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}})$. □

The following example shows that the Weil logarithmic height function is strictly positive on $\mathbb{P}^n_{\mathbb{Q}}$.

Example 2. *Consider the weights $\mathfrak{q} = (1, \ldots, 1)$. Then $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} = \mathbb{P}^n_{\mathbb{Q}}$ is the projective space and the weighted height $\mathcal{S}_{\mathbb{Q}}$ is simply the projective height $H_{\mathbb{Q}}$. Since $m = \mathrm{lcm}(q_0, \ldots, q_n) = 1$ then there are no primes dividing m and $\mathrm{Sing}\,\mathbb{P}^n_{\mathbb{Q}} = \emptyset$. On the other side from (31) we have*

$$\begin{aligned}
\log \mathrm{h}_{\mathrm{wgcd}}(\mathbf{x}) &= \sum_{\nu_p \in M_{\mathbb{Q}}} \min\left\{ \left\lfloor \frac{\nu_p^+(x_0)}{q_0} \right\rfloor, \ldots, \left\lfloor \frac{\nu_p^+(x_n)}{q_n} \right\rfloor \right\} \\
&= \sum_{\nu_p \in M_{\mathbb{Q}}} \min\left\{\nu_p^+(x_0), \ldots, \nu_p^+(x_n)\right\} \\
&\geq \min\{\left\{\nu_\infty^+(x_0), \ldots, \nu_\infty^+(x_n)\right\} > 0,
\end{aligned}$$

since at least one of the coordinates $x_i \neq 0$,

Lemma 2. *Let $\mathcal{X}$ be a smooth weighted variety defined over $\mathbb{Q}$, and $\mathcal{Y}$ a subvariety of $\mathcal{X}$ of codimension $r \geq 2$. Let $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{Y}$ and denote by $\tilde{\mathcal{Y}} := \pi^{-1}(\mathcal{Y})$ its the exceptional divisor. Then,*

(i) *$\pi\,|_{\pi^{-1}(\mathcal{X}\setminus\mathcal{Y})} : \pi^{-1}(\mathcal{X}\setminus\mathcal{Y}) \to \mathcal{X}\setminus\mathcal{Y}$ is an isomorphism.*

(ii) *Exceptional divisor $\tilde{\mathcal{Y}}$ is an effective Cartier divisor on $\tilde{\mathcal{X}}$.*

Proof. This is a direct consequence of [8, Prop. II.7.13]. For every $y \in \mathcal{Y}$ we have an open neighborhood $\mathcal{U}$ around $\pi^{-1}(y)$ and $f \in \mathcal{U}$. The conditions from the definition of Cartier divisors are satisfied. □

Proposition 3. *Let $\mathcal{X} := \prod_{i=1}^{n} \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}}$, and consider $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$, the blow-up of $\mathcal{X}$ along $\bar{0} = ([1:0],[1:0],\cdots,[1:0])$. Denote by $\tilde{\mathcal{Y}} = \pi^{-1}(\bar{0})$ the exceptional divisor of this blow-up. Then, for all $\nu \in M_{\mathbb{Q}}$ and any non-singular points*

$$\mathbf{x} = [x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \setminus \{[1:0:\cdots:0]\}$$

with $\bar{x} = \varphi_{n,\mathfrak{q}}\,|_{\mathcal{X}}\,(\mathbf{x}) \in \mathcal{X}(\mathbb{Q}) \setminus \{\bar{0}\}$, we have

$$\zeta_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\pi^{-1}(\bar{x}),\nu) = \min\left\{\left\lfloor \frac{\nu^+(x_0)}{q_0}\right\rfloor,\ldots,\left\lfloor \frac{\nu^+(x_n)}{q_n}\right\rfloor\right\} = T_\nu(\bar{x}), \tag{34}$$

and

$$\log \mathrm{h}_{\mathrm{wgcd}}\,(\mathbf{x}) = \sum_{\nu\in M_{\mathbb{Q}}} \zeta_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\pi^{-1}(\bar{x}),\nu) = \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}\left(\pi^{-1}(\bar{x}),\nu\right). \tag{35}$$

Proof. By Lem. 2, $\tilde{\mathcal{Y}}$ is an effective divisor on $\tilde{\mathcal{X}}$. Hence, using the functoriality of local weighted heights, we have

$$\begin{aligned}
\zeta_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\pi^{-1}(\bar{x}),\nu) = \zeta_{\mathcal{X},\bar{0}}(\bar{x},\nu) &= \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},[1:0:\cdots:0]}\left([x_0^{\frac{1}{q_0}} : x_1^{\frac{1}{q_1}} : \cdots : x_n^{\frac{1}{q_n}}],\nu\right),\\
&= \min\left\{\nu^+(x_0^{\frac{1}{q_1}}),\cdots,\nu^+(x_n^{\frac{1}{q_n}})\right\}\\
&= \min\left\{\left\lfloor \frac{\nu^+(x_0)}{q_0}\right\rfloor,\cdots,\left\lfloor \frac{\nu^+(x_n)}{q_n}\right\rfloor\right\} = T_\nu(\bar{x}).
\end{aligned}$$

Adding these weighted local heights together we get the global formula.

□

The above result leads to the following definition.

Definition 1. *Let $\mathcal{X}$ be a smooth weighted variety defined over $\mathbb{Q}$, and $\mathcal{Y}$ a subvariety of $\mathcal{X}$ of codimension $r \geq 2$ and $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$, the blow-up of $\mathcal{X}$ along $\mathcal{Y}$. For any $P \in (\mathcal{X}\setminus\mathcal{Y})(\mathbb{Q})$ we denote by*

$$\tilde{P} := \pi^{-1}(P) \in \tilde{\mathcal{X}} \quad \text{and} \quad \tilde{\mathcal{Y}} = \pi^{-1}(\mathcal{Y}).$$

The ***generalized logarithmic weighted greatest common divisor of the point*** P ***with respect to*** $\mathcal{Y}$ *is defined to be*

$$\log \mathrm{h}_{\mathrm{wgcd}}\left(P;\mathcal{Y}\right)=\mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}). \tag{36}$$

A point $\mathbf{x}=[x_0:\cdots:x_n]\in\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is called *normalized* if it has integers coordinates and $\mathrm{wgcd}\left(x_0,x_1,\cdots,x_n\right)=1$; see [2] for details.

Lemma 3. *Let* $\mathfrak{q}=(q_0,\cdots,q_n)$ *be well-formed weights,* $m=\mathrm{lcm}(q_0,\cdots,q_n)$, $\mathbf{y}=[1:0:\cdots:0]$, *and* $\mathbf{x}=[x_0:\cdots:x_n]\in\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ *a smooth and normalized point. Then*

$$\log \mathrm{h}_{\mathrm{wgcd}}\left(\mathbf{x};\{\mathbf{y}\}\right)=\log \gcd(x_1,\ldots,x_n)+O(1). \tag{37}$$

Proof. Indeed, letting $\mathfrak{q}_i=(q_0,q_i)$ for each $i=1,\cdots,n$ and

$$\mathcal{X}=\prod_{i=0}^{n}\mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}},$$

then considering the rational map $\pi_{n,\mathfrak{q}}:\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}\to\mathcal{X}$, we have $\bar{0}=\varphi_{n,\mathfrak{q}}(\mathbf{y})$, where

$$\bar{0}=([1:0],\cdots,[1:0])\in\mathcal{X}.$$

Let $\bar{x}=\varphi_{n,\mathfrak{q}}(\mathbf{x})$ and apply Prop. 3, to the blow-up $\pi:\tilde{\mathcal{X}}\to\mathcal{X}$ along $\mathcal{Y}=\{\bar{0}\}$. Let $\tilde{\mathcal{Y}}=\pi^{-1}(\mathcal{Y})$ be the exceptional divisor of the blow-up. Then

$$\log \mathrm{h}_{\mathrm{wgcd}}\left(\mathbf{x};\{\mathbf{y}\}\right)=\log \mathrm{h}_{\mathrm{wgcd}}\left(\bar{x};\mathcal{Y}\right)=\mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}\left(\pi^{-1}(\bar{x})\right)$$

By definition of the global weighted height and properties of local weighted height [13, Thm. 1 (iv)], one can see that the last term is equal to the right-hand side of (37). □

One can extend the result of Lem. 3, as stated in following proposition.

Proposition 4. *Let* $\mathfrak{q}=(q_0,\cdots,q_n)$ *be a well-formed set of weights and* $m=\mathrm{lcm}(q_0,\cdots,q_n)$. *Assume that* $\mathcal{Z}\subset\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ *is a closed subvariety defined by the weighted homogeneous polynomials* $f_1,\cdots,f_r$ *with integer coefficients such that*

$$\mathcal{Z}\cap\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}})=\emptyset.$$

Then

$$\log \mathrm{h}_{\mathrm{wgcd}}\left(\mathbf{x};\mathcal{Z}\right)=\log \gcd(f_1(\mathbf{x}),\ldots,f_r(\mathbf{x}))+O(1), \tag{38}$$

for $\mathbf{x}\in\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}\backslash\left\{\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}})\cup\mathrm{Supp}(\mathcal{Z})\right\}$ *with* $x_i\in\mathbb{Z}$ *and* $\mathrm{wgcd}\left(x_0,x_1,\cdots,x_n\right)=1$.

Proof. Let $\mathcal{Y}$ be given by

$$\mathcal{Y}=\varphi_{n,\mathfrak{q}}(\mathcal{Z})\subset\mathcal{X}=\prod_{i=1}^{n}\mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}}, \tag{39}$$

where $\mathfrak{q}_i = (q_0, q_i)$ for $i = 1, \cdots, n$ and where $\varphi_{n,\mathfrak{q}}$ is defined by (26).

Consider the blow-up $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ along $\mathcal{Y}$ and its exceptional divisor $\tilde{\mathcal{Y}} = \pi^{-1}(\mathcal{Y})$. Let $\mathbf{y} = \varphi_{n,\mathfrak{q}}(\mathbf{x})$, for any $\mathbf{x} \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \setminus \left\{ \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) \cup \mathrm{Supp}(\mathcal{Z}) \right\}$. Then, we have

$$\begin{aligned}
\log \mathrm{h}_{\mathrm{wgcd}}\left(\mathbf{x}; \mathcal{Z}\right) &= \sum_{\nu_p \in M_{\mathbb{Q}}} \zeta_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\pi^{-1}(\mathbf{y}), \nu_p) \\
&= \sum_{\nu_p \in M_{\mathbb{Q}}} \zeta_{\mathcal{X},\mathcal{Y}}(\mathbf{y}, \nu_p) \\
&= \sum_{\nu_p \in M_{\mathbb{Q}}} \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_p). \\
&= \sum_{p \in M_{\mathbb{Q}} \setminus \{\infty\}} \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_p) + \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_\infty)
\end{aligned}$$

By definition (see [13, Eq. (37)]) of the local weighted height associated to $\mathcal{Z}$, for $\nu_p \in M_{\mathbb{Q}}$, we have

$$\begin{aligned}
\zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_p) &= \min_{1 \le j \le r} \left\{ -\log \frac{|f_j(\mathbf{x})|_{\nu_p}}{\max_i |x_i|_{\nu_p}^{\frac{m}{q_i}}} \right\} \\
&= \log \max_{1 \le j \le r} |f_j(\mathbf{x})|_{\nu_p} + \frac{m}{q_i} \log \max_i |x_i|_{\nu_p}.
\end{aligned}$$

Thus,

$$\sum_{\nu_p \in M_{\mathbb{Q}}} \zeta_{\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}},\mathcal{Z}}(\mathbf{x}, \nu_p) = \log \gcd(f_1(\mathbf{x}), \ldots, f_t(\mathbf{x})) + O(1).$$

Therefore, putting all together gives the desired equality.

$$\log \mathrm{h}_{\mathrm{wgcd}}\left(\mathbf{x}; \mathcal{Z}\right) = \log \gcd(f_1(\mathbf{x}), \ldots, f_t(\mathbf{x})) + O(1).$$

This completes the proof. □

6 Vojta's conjecture and bounds on greatest common divisors

Let $\mathcal{X}$ be a smooth weighted variety defined over $\mathbb{Q}$, and $\mathcal{Y}$ a subvariety of $\mathcal{X}$ of codimension $r \ge 2$ and $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$, the blow-up of $\mathcal{X}$ along $\mathcal{Y}$. It is a well-known fact that the canonical bundle of the blow-up $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ along the subvariety $\mathcal{Y} \subset \mathcal{X}$ is given by

$$K_{\tilde{\mathcal{X}}} \sim \pi^\star K_{\mathcal{X}} + (r-1)\tilde{\mathcal{Y}},$$

where $\tilde{\mathcal{Y}} = \pi^{-1}(\mathcal{Y})$. If $\mathcal{A}$ is an ample divisor on $\mathcal{X}$, then there is an integer N such that $-\tilde{\mathcal{Y}} + N\pi^\star\mathcal{A}$ is ample on $\tilde{\mathcal{X}}$, see [8, Thm. A.5.1]. Let

$$\tilde{\mathcal{A}} = -\frac{1}{N}\tilde{\mathcal{Y}} + \pi^\star\mathcal{A}$$

be the ample cone of $\tilde{\mathcal{X}}$. We further assume that $-K_{\mathcal{X}}$ is a normal crossing and

$$\mathrm{Supp}(K_{\mathcal{X}}) \cap \mathcal{Y} = \emptyset.$$

Let S be a finite set of places of $\mathbb{Q}$ and define

$$\mathfrak{s}_{\mathcal{X},D,S}(\cdot) := \sum_{\nu \in S} \zeta_{\mathcal{X},D}(\cdot,\nu) \quad \text{and} \quad \mathfrak{s}'_{\mathcal{X},D,S}(\cdot) := \sum_{\nu \notin S} \zeta_{\mathcal{X},D}(\cdot,\nu) \tag{40}$$

Then we have the following:

Theorem 1. *Let $\mathcal{X}$ be a smooth projective weighted variety, $\mathcal{A}$ an ample divisor on $\mathcal{X}$, $\mathcal{Y} \subset \mathcal{X}$ a smooth subvariety of codimension $r \geq 2$, and $-K_{\mathcal{X}}$ a normal crossing divisor whose support does not intersect $\mathcal{Y}$, all defined over $\mathbb{Q}$. Assume Vojta's conjecture (see Conj. 4) for smooth weighted varieties. Then for every finite set of places S containing $M_{\mathbb{Q}}^{\infty}$ and every $0 < \varepsilon < r-1$ there is a proper closed subvariety $\mathcal{Z} = \mathcal{Z}(\varepsilon, \mathcal{X}, \mathcal{Y}, \mathcal{A}, S)$ of $\mathcal{X}$, and constants $C_\varepsilon = C_\varepsilon(\mathcal{X}, \mathcal{Y}, \mathcal{A}, S)$ and $\delta_\varepsilon = \delta_\varepsilon(\mathcal{X}, \mathcal{Y}, \mathcal{A})$, such that*

$$\log \mathrm{h}_{\mathrm{wgcd}}(P; \mathcal{Y}) \leq \varepsilon\, \mathfrak{s}_{\mathcal{X},\mathcal{A}}(P) + \frac{1}{r-1+\delta_\varepsilon}\, \mathfrak{s}'_{\mathcal{X},-K_{\mathcal{X}},S}(P) + C_\varepsilon, \tag{41}$$

for all $P \in (\mathcal{X} \setminus \mathcal{Z})(\mathbb{Q})$.

Proof. We apply Conj. 4, for the weighted blow-up $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ and the divisor $D = -\pi^\star K_{\mathcal{X}}$ to get

$$\mathfrak{s}_{\tilde{\mathcal{X}},-\pi^\star K_{\mathcal{X}},S}(\tilde{P}) + \mathfrak{s}_{\tilde{\mathcal{X}},K_{\mathcal{X}}}(\tilde{P}) \leq \varepsilon\, \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{A}}}(\tilde{P}) + C_\varepsilon,$$

for all $\tilde{P} \in (\tilde{\mathcal{X}} \setminus \tilde{\mathcal{Z}})(\mathbb{Q})$. Substituting $K_{\tilde{\mathcal{X}}} = \pi^\star K_{\mathcal{X}} + (r-1)\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{A}} = -\frac{1}{N}\tilde{\mathcal{Y}} + \pi^\star \mathcal{A}$ we get

$$-\mathfrak{s}_{\mathcal{X},K_{\mathcal{X}},S}(P) + \mathfrak{s}_{\mathcal{X},K_{\mathcal{X}}}(P) + (r-1)\,\mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) \leq \varepsilon\, \mathfrak{s}_{\mathcal{X},\mathcal{A}}(P) - \frac{\varepsilon}{N}\, \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) + C_\varepsilon$$

for all $P \in (\mathcal{X} \setminus \pi(\tilde{\mathcal{Z}}))(\mathbb{Q})$. Since $-\mathfrak{s}_{\mathcal{X},K_{\mathcal{X}},S}(P) + \mathfrak{s}_{\mathcal{X},K_{\mathcal{X}}}(P) = \mathfrak{s}'_{\mathcal{X},K_{\mathcal{X}},S}$ we have

$$\mathfrak{s}'_{\mathcal{X},K_{\mathcal{X}},S}(P) + \left(r - 1 + \frac{\varepsilon}{N}\right) \mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) \leq \varepsilon\, \mathfrak{s}_{\mathcal{X},\mathcal{A}}(P) + C_\varepsilon,$$

for all $P \in (\mathcal{X} \setminus \mathcal{Z})(\mathbb{Q})$. Hence,

$$\mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) \leq \frac{N}{N(r-1)+\varepsilon} \left(-\mathfrak{s}'_{\mathcal{X},K_{\mathcal{X}},S}(P) + \varepsilon\, \mathfrak{s}_{\tilde{\mathcal{X}},\mathcal{A}}(P) + C_\varepsilon \right)$$

Since $\mathfrak{s}_{\tilde{\mathcal{X}},\tilde{\mathcal{Y}}}(\tilde{P}) = \log \mathrm{h}_{\mathrm{wgcd}}(P; \mathcal{Y})$, we have

$$\log \mathrm{h}_{\mathrm{wgcd}}(P; \mathcal{Y}) \leq \frac{N}{N(r-1)+\varepsilon} \left(-\mathfrak{s}'_{\mathcal{X},K_{\mathcal{X}},S}(P) + \varepsilon\, \mathfrak{s}_{\tilde{\mathcal{X}},\mathcal{A}}(P) + C_\varepsilon \right).$$

Finally, setting $\delta = \varepsilon/N$ gives (41) and this completes the proof.

□

Let $\mathfrak{q} = (q_0, \cdots, q_n)$ and assume that $X = \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is well-formed weighted projective space. For $i = 0, \ldots, n$, let

$$H_i = \{\mathbf{x} \in X \mid x_i = 0\},$$

$A_0 = H_0$, and $K_X = -\sum_{i=0}^n H_i$. the map

$$\varphi_{n,\mathfrak{q}} : \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \to \mathcal{X} := \prod_{i=0}^n \mathbb{P}^1_{\mathfrak{q}_i,\mathbb{Q}},$$

where $\mathfrak{q}_i = (q_0, q_i)$ for $i = 1, \cdots, n$ and denote by

$$\mathcal{H}_i = \varphi_{n,\mathfrak{q}}(H_i), \quad \mathcal{A}_0 = \varphi_{n,\mathfrak{q}}(H_0), \quad K_{\mathcal{X}} = -\sum_{i=0}^n \varphi_{n,\mathfrak{q}}(H_i).$$

Notice that the canonical divisor $K_{\mathcal{X}}$ is a normal crossing on $\mathcal{X}$ satisfying

$$\mathcal{Y} \cap \mathrm{Supp}(-K_{\mathcal{X}}) = \emptyset, \qquad \text{where } \mathcal{Y} = \varphi_{n,\mathfrak{q}}(\mathcal{Z}).$$

Let S be a finite set of primes. The "prime-to-S" part $|x|'_S$ of any nonzero integer x is defined to be the largest divisor of x that is not divisible by any of the primes in S, in other words,

$$|x|'_S = |x| \cdot \prod_{p \in S} |x|_p.$$

In particular, x is an S-unit if and only if $|x|'_S = 1$.

Theorem 2. *Keeping the above notations in mine, we assume that $\mathcal{Z} \subset \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ is a closed subvariety defined by*

$$f_1, \cdots, f_t \in \mathbb{Z}_{\mathfrak{q}}[x_0, \ldots, x_n],$$

of codimension $r := n - \dim(\mathbb{Z}) \geq 2$ in $\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$ such that $\mathcal{Z} \cap \mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) = \emptyset$, and has transversal intersection with the union $\cup_{i=0}^n \mathcal{H}_i$. Let S be a finite set of primes and $\varepsilon > 0$. If Conj. 4, holds for the weighted blow-up $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ along $\mathcal{Y}$, then there exists a nonzero weighted polynomial $g \in \mathbb{Z}_{\mathfrak{q}}[x_0, \ldots, x_n]$ and a constant $\delta = \delta_{\varepsilon,\mathcal{Z}} > 0$, such that every $\tilde{\alpha} = (\alpha_0, \cdots, \alpha_n) \in \mathbb{Z}^{n+1}$ with $\mathrm{wgcd}\,(\alpha_0, \cdots, \alpha_n) = 1$ satisfies either $g(\tilde{\alpha}) = 0$ or

$$\gcd(f_1(\tilde{\alpha}), \cdots, f_t(\tilde{\alpha})) \leq \max\left\{|\alpha_0|^{\frac{1}{q_0}}, \cdots, |\alpha_n|^{\frac{1}{q_n}}\right\}^{\varepsilon} \cdot (|\alpha_0 \cdots \alpha_n|'_S)^{\frac{1}{m(r-1+\delta)}}, \quad (42)$$

where $|\cdot|'_S$ is the "prime-to-S" part of its origin.

Proof. By definition of the global weighted height for

$$\mathbf{x} = [\alpha_0 : \cdots : \alpha_n] \in \mathbb{P}^n_{\mathfrak{q},\mathbb{Q}} \backslash \{\mathrm{Sing}(\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}) \cup \mathrm{Supp}(\mathcal{Z})\}(\mathbb{Q})$$

with $\mathrm{wgcd}\,(\alpha_0,\cdots,\alpha_n)=1$, we have

$$\mathfrak{s}_{X,A}(\mathbf{x}) = \log\,\max\{|\alpha_0|^{\frac{1}{q_0}},\cdots,|\alpha_n|^{\frac{1}{q_n}}\} + O(1) \tag{43}$$

By Prop. 4, we have

$$\log \mathrm{h}_{\mathrm{wgcd}}\,(\mathbf{x};\mathcal{Z}) = \log \gcd(f_1(\mathbf{x}),\cdots,f_t(\mathbf{x})) + O(1). \tag{44}$$

Let $\mathbf{y}=\varphi_{n,\mathfrak{q}}(\mathbf{x})$ for $\mathbf{x}=[\alpha_0:\cdots:\alpha_n]\in\mathbb{P}^n_{\mathfrak{q},\mathbb{Q}}$. Then, by definition of S-part of the weighted heights and functoriality of the weighted heights, we have

$$\mathfrak{s}'_{\mathcal{X},\mathcal{H}_i,S}(\mathbf{y}) = \mathfrak{s}'_{X,H_i,S}(\mathbf{x}) = \sum_{\nu\in S}\nu^+(\alpha_i) = \frac{1}{q_i}\log|\alpha_i|'_S,$$

which implies that

$$\mathfrak{s}'_{\mathcal{X},-K_{\mathcal{X}},S}(\mathbf{y}) = \mathfrak{s}'_{X,-K_X,S}(\mathbf{x}) = \sum_{i=0}^{n}\mathfrak{s}'_{X,H_i,S} = \frac{1}{m}\log|\alpha_0\alpha_1\cdots\alpha_n|'_S. \tag{45}$$

By substituting (43), (44), (45), in (41), one obtains

$$\begin{aligned}
\log\gcd(f_1(\tilde{\alpha}),\cdots,f_t(\tilde{\alpha})) &\le \log \mathrm{h}_{\mathrm{wgcd}}\,(\mathbf{x};\mathcal{Z}) = \log \mathrm{h}_{\mathrm{wgcd}}\,(\mathbf{y};\mathcal{Y})\\
&\le \varepsilon\,\mathfrak{s}_{\mathcal{X},\mathcal{A}}(\mathbf{y}) + \frac{1}{r-1+\delta}\,\mathfrak{s}'_{\mathcal{X},-K_{\mathcal{X}},S}(\mathbf{y}) + C_\varepsilon,\\
&\le \varepsilon\cdot\log\max\{|\alpha_0|^{\frac{1}{q_0}},\cdots,|\alpha_n|^{\frac{1}{q_n}}\}\\
&\quad\cdot\frac{1}{m(r-1+\delta)}\log\left(|\alpha_0\alpha_1\cdots\alpha_n|'_S\right) + C_\varepsilon,
\end{aligned}$$

where $\delta=\delta_{\varepsilon,\mathcal{Z}}$. Then, by exponentiation, we obtain the desired inequality (2). □

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