RATIONAL FUNCTIONS ON THE PROJECTIVE LINE FROM A COMPUTATIONAL VIEWPOINT

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Abstract. We develop an explicit invariant-theoretic description of the moduli space $\mathcal{M}_3^1$ of degree-three rational maps on $\mathbb{P}^1$. Using the Clebsch–Gordan decomposition, a cubic map ϕ is represented, up to conjugation, by a pair of binary forms $(f, g) \in V_4 \oplus V_2$. This identification allows us to construct a system of weighted projective invariants $\xi_0, \ldots, \xi_5$ of degrees $2, 2, 3, 3, 4, 6$, yielding an embedding of $\mathcal{M}_3^1$ into $\mathbb{P}^5_{(2,2,3,3,4,6)}$ on the locus where the nonzero coordinates have coprime weights. We further introduce absolute invariants, defined as weight-zero rational functions of the ξ_i normalized by an invariant I_6, and prove that they uniquely determine the isomorphism class.

We determine the algebraic relations among the invariants and obtain a complete stratification of $\mathcal{M}_3^1$ by automorphism groups, allowing the automorphism group of any rational cubic to be recovered directly from its invariants.

1. Introduction

Let k be an algebraically closed field of characteristic zero and $\mathbb{P}^1_k$ the projective line over k. A degree $d \geq 2$ rational function $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is given as the ratio of two degree d binary forms, say $\phi(x, y) = \frac{f_0(x,y)}{f_1(x,y)}$ such that the resultant between $f_0(x, y)$ and $f_1(x, y)$ is non-zero. Hence, a rational function is a pair of binary forms of the same degree with no common roots. If we denote $f_0(x, y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$ and $f_1(x, y) = \sum_{i=0}^{d} b_i x^i y^{d-i}$, then the collection of pairs $[f_0 : f_1]$ can be parametrized via $[a_d : \cdots : a_0 : b_d : \cdots : b_0] \in \mathbb{P}^{2d+1}$, such that $\mathrm{Res}(f_0, f_1) \neq 0$. So the parameter space of degree $d > 1$ rational functions on $\mathbb{P}^1$ is the complement of the resultant locus in $\mathbb{P}^{2d+1}$, say $\mathrm{Rat}_d^1 := \mathbb{P}^{2d+1} \setminus V(\mathrm{Res})$.

The group $\mathrm{SL}_2(k)$ acts on Rat_d^1 by conjugation, i.e., for some $M \in \mathrm{SL}_2(k)$, $\phi \to \phi^M := M^{-1} \circ \phi \circ M$. Two rational functions $\phi, \psi \in \mathrm{Rat}_d^1$ are called **conjugate** if there is an $M \in \mathrm{SL}_2(k)$ such that $\phi = \psi^M$. The moduli space of degree $d > 1$ rational functions (in one variable) is denoted by $\mathcal{M}_d^1$ and can be constructed as a quotient space of this SL_2-action.

The automorphism group of ϕ is defined as

$$\mathrm{Aut}(\phi) := \{\sigma \in \mathrm{PGL}_2(k) \mid \phi^\sigma = \phi\}.$$

It is a finite subgroup of $\mathrm{PGL}_2(k)$, so it is isomorphic to one of the following: a cyclic group C_n, a dihedral group D_n, A_4, S_4, or A_5. Determining which one of these groups occur for a fixed degree $d \geq 2$ is studied in [5, 7, 10].

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For any $\phi(x,y) = \frac{f_0(x,y)}{f_1(x,y)}$, we define degree $(d+1)$ and $(d-1)$ binary forms as

$$\mathcal{I}_\phi := yf_0 - xf_1 \quad \text{and} \quad \mathcal{J}_\phi := \frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}.$$

Any two degree d rational functions ϕ and ψ are conjugate for some $M \in \mathrm{PGL}_2(k)$ via $\psi = \phi^M$ if and only if $\mathcal{I}_\psi = \mathcal{I}_\phi^M$ and $\mathcal{J}_\psi = \mathcal{J}_\phi^M$; see Lem. 1. Moreover, there is a one-to-one correspondence between degree d rational functions $\phi(x)$ and points $(f,g) \in V_{d+1} \oplus V_{d-1}$ such that

$$\mathrm{Res}\left(xg + \frac{\partial f}{\partial y}, yg - \frac{\partial f}{\partial x}\right) \neq 0.$$

Hence, determining invariants of rational functions is the same as determining generators for the ring of invariants of $V_{d+1} \oplus V_{d-1}$, which can be determined using a result of Clebsch once generators of the ring of invariants for V_{d+1} and V_{d-1} are known.

Denote by $\mathcal{R}_{(d+1,d-1)}$ the ring of invariants of $V_{d+1} \oplus V_{d-1}$ and $(\xi_0, \ldots, \xi_n)$ the tuple of generators of this ring with degrees $(q_0, \ldots, q_n)$ respectively. Since all $\xi_0, \ldots, \xi_n$ are homogeneous polynomials, then $\mathcal{R}_{(d+1,d-1)}$ is a graded ring and $\mathrm{Proj}\,\mathcal{R}_{(d+1,d-1)}$ is a weighted projective space denoted by $\mathbb{P}^n_\omega(k)$, for some set of weights where $\omega = (q_0, \ldots, q_n)$. Thus for each $\phi \in \mathcal{P}_d$, we evaluate its invariants and have a map $\xi : \mathrm{Rat}^1_d \to \mathbb{P}^n_\omega$ via

$$\phi \to [\xi_0(\phi) : \cdots : \xi_n(\phi)].$$

Determining automorphism groups, describing the inclusions among their loci, and analyzing fields of definition becomes increasingly subtle as the degree d increases; see [18]. The invariant-theoretic description of $\mathcal{M}^1_d$ grows rapidly in complexity, and for $d > 3$ the structure of the rings $\mathcal{R}_{(d+1,d-1)}$ is largely unknown.

This setting is closely analogous to the moduli theory of hyperelliptic and superelliptic curves, where explicit invariants play a central role in making the geometry computationally accessible. In the case $d = 3$, the invariant theory is fully explicit. For the representation $V_4 \oplus V_2$ the ring of invariants is generated by six homogeneous polynomials of degrees $2, 2, 3, 3, 4$, and 6, giving an embedding

$$\mathcal{M}^1_3 \hookrightarrow \mathbb{P}^5_{(2,2,3,3,4,6)}$$

onto the locus where the gcd of the weights of the non-zero coordinates equals 1 (see Prop. 1). We compute these generators and determine the loci corresponding to each finite automorphism group, obtaining equations for the strata with groups C_2, C_3, C_4, V_4, D_4, and A_4. As an arithmetic application, we enumerate rational cubics over $\mathbb{Q}$ of naive height ≤ 4 and study their images in the weighted projective moduli space. Working in $\mathcal{M}^1_3$ removes the redundancies inherent in the coefficient space and allows the automorphism strata to be described cleanly and explicitly.

Although the case $d = 3$ serves as a complete model, extending this framework to $d > 3$ remains an open problem, chiefly because the generators of $\mathcal{R}_{(d+1,d-1)}$ are unknown. The present work illustrates how an explicit invariant-theoretic description of $\mathcal{M}^1_d$ enables a systematic analysis of automorphism loci and suggests a path forward for higher degrees.

The code used to generate the dataset, compute the invariants ξ_i, and perform the machine learning experiments, together with the full dataset, is available at the GitHub repository [17].

2. Preliminaries

Let k be an algebraically closed field, $\mathbb{P}^N(k)$ the projective N-space over k, and $k[x,y]$ be the polynomial ring in two variables. Throughout this paper we assume $\mathrm{char}(k)=0$. By V_d we denote the $(d+1)$-dimensional subspace of $k[x,y]$ consisting of homogeneous polynomials

$$f(x,y)=a_dx^d+a_{d-1}x^{d-1}y+\cdots a_1xy^{d-1}+a_0y^d,$$

of degree d (up to multiplication by a scalar). Elements in V_d (up to multiplication by a constant) are called **binary forms** of degree d. $\mathrm{GL}_2(k)$ acts as a group of automorphisms on $k[x,y]$ as follows:

$$M=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in \mathrm{GL}_2(k),\ \text{then}\quad M\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}ax+by\\ cx+dy\end{bmatrix} \tag{1}$$

Denote by f^M the binary form $f^M(x,y):=f(ax+by,cx+dy)$. It is well known that while $\mathrm{SL}_2(k)$ does not fix a binary form, it preserves its equivalence class under the natural change-of-variables action.

Consider a_0, a_1, ... , a_d as parameters (coordinate functions on V_d). Then the coordinate ring of V_d can be identified with $k[a_0,\dots,a_d]$. For $I\in k[a_0,\dots,a_d]$ and $M\in\mathrm{GL}_2(k)$, define $I^M\in k[a_0,\dots,a_d]$ as $I^M(f):=I(f^M)$, for all $f\in V_d$. Then $I^{MN}=(I^M)^N$ and $I^M(f)$ define an action of $\mathrm{GL}_2(k)$ on $k[a_0,\dots,a_d]$.

A homogeneous polynomial $I\in k[a_0,\dots,a_d,x,y]$ is called a **covariant** of index s if $I^M(f)=\delta^sI(f)$, for all $f\in V_d$, where $\delta=\det(M)$. The homogeneous degree in $a_0,\dots,a_d$ is called the **degree** of I, and the homogeneous degree in x,y is called the **order** of I. A covariant of order zero is called **invariant**. An invariant is a $\mathrm{SL}_2(k)$-invariant on V_d.

By Hilbert's basis theorem the ring of invariants of binary forms is finitely generated. We denote by $\mathcal{R}_d$ the ring of invariants of the binary forms of degree d. Then, $\mathcal{R}_d$ is a finitely generated graded ring; see [9] for details among many other sources.

2.1. **Change of coordinates.** Let $I_0,\dots,I_n$ be the generators of $\mathcal{R}_d$ with degrees $q_0,\dots,q_n$ respectively. For any two binary forms f and g, $f=g^M$, $M\in\mathrm{GL}_2(k)$, if and only if

$$(I_0(f),\dots I_i(f),\dots,I_n(f))=(\lambda^{q_0}I_0(g),\dots,\lambda^{q_i}I_i(g),\dots,\lambda^{q_n}I_n(g)), \tag{2}$$

where $\lambda=(\det M)^{\frac{d}{2}}$.

2.2. **Generators of the ring of invariants.** Let V_d be the space of degree $d>1$ binary forms defined over k, and $\mathcal{R}_d$ the ring of invariants. Below we list the generating set of $\mathcal{R}_d$ for $d\le 10$. We assume that the binary forms are given in standard form $f(x,y)=\sum_{i=0}^d\binom{d}{i}a_ix^iy^{d-i}$. For $f,g\in V_d$ the **r-th transvectant** of f and g is defined as

$$(f,g)_r:=\frac{(m-r)!\,(n-r)!}{n!\,m!}\sum_{k=0}^{r}(-1)^k\begin{bmatrix}r\\ k\end{bmatrix}\cdot\frac{\partial^rf}{\partial x^{r-k}\,\partial y^k}\cdot\frac{\partial^rg}{\partial x^k\,\partial y^{r-k}},$$

While there is no method known to determine a minimal generating set of invariants for any $\mathcal{R}_d$, we display such sets for $d=3,4$ an their degree for $d\le 10$ as in [4].

Example 1 (Cubics)**.** *A generating set for $\mathcal{R}_3$ is $\xi=[\xi_0]$, where*

$$\xi_0=((f,f)_2,(f,f)_2)_2=-54a_0^2a_3^2+36a_1a_3a_0a_2-8a_2^3a_0-8a_1^3a_3+2a_2^2a_1^2 \tag{3}$$

Example 2 (Quartics)**.** *A generating set for $\mathcal{R}_4$ is $\xi = [\xi_0, \xi_1]$ with $\omega = (2, 3)$ is the weight of the generating set, where $\xi_0 = (f, f)_4$ and $\xi_1 = (f, (f, f)_2)_4$.*

3. Invariants of Rational Functions

This section derives the invariants of degree $d > 1$ rational functions on $\mathbb{P}^1(k)$, where k is algebraically closed. Our goal is to classify these functions up to $\mathrm{PGL}_2(k)$-equivalence, compute the invariant ring $\mathcal{R}_{(d+1,d-1)}$, and specialize to $d = 3$ for explicit invariants that enable machine learning analysis of the moduli space $\mathcal{M}_3^1$.

If one fixes homogeneous coordinates x, y on $\mathbb{P}^1$, then any rational function $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$ can be realized as

$$\phi(x, y) = \frac{f_0(x, y)}{f_1(x, y)}, \tag{4}$$

where $d = \max\{\deg f_0, \deg f_1\}$ and f_0 and f_1 are homogeneous polynomials of degree at most d. Conversely, a pair of homogeneous polynomials of degree d in x and y determine a rational function

$$\phi(x, y) = [f_0(x, y) : f_1(x, y)] \tag{5}$$

if f_0, f_1 have no common roots in k. For a fixed degree $d > 1$, the collection of all such pairs of homogeneous polynomials $[f_0(x, y) : f_1(x, y)]$, say

$$f_0 = \sum_{i=0}^{d} a_i x^{d-i} y^i \quad \text{and} \quad f_1 = \sum_{i=0}^{d} b_i x^{d-i} y^i, \tag{6}$$

can be naturally parametrized as the projective space $\mathbb{P}^{2d+1}$, via

$$[f_0 : f_1] \to [a_0 : a_1 : \cdots : a_d : b_0 : \cdots : b_d] \in \mathbb{P}^{2d+1}.$$

We denote the resultant of two binary forms f_0 and f_1 by $\mathrm{Res}(f_0, f_1)$. Notice that it's well-defined and a degree $2d$ homogeneous polynomial in

$$I_{2d}(\phi) := \mathrm{Res}(f_0, f_1) \in k[a_0, \ldots, a_d, b_0, \ldots, b_d]. \tag{7}$$

Hence, $I_{2d}(\phi)$ is an $\mathrm{SL}_2(k)$-invariant of degree $2d$. Moreover, ϕ is a rational function on $\mathbb{P}^1$ if and only if $I_{2d}(\phi) \neq 0$.

We can construct the parameter space of rational functions on $\mathbb{P}^1$ as the complement of the vanishing locus $V(I_{2d})$ of I_{2d}. Hence the **space of rational functions of degree d on $\mathbb{P}^1$** is defined as

$$\mathrm{Rat}_d^1 := \mathbb{P}^{2d+1} \setminus V(I_{2d}).$$

The action of $\mathrm{PGL}_2(k)$ on V_d extends naturally to an action on Rat_d^1. For each $\sigma \in \mathrm{PGL}_2(k)$, we have $\mathrm{PGL}_2(k) \times \mathrm{Rat}_d^1 \to \mathrm{Rat}_d^1$ via

$$(\sigma, \phi(x, y)) \to \phi^\sigma := \sigma^{-1} \phi \sigma,$$

Two rational functions $\phi, \psi \in \mathrm{Rat}_d^1$ are called **k-conjugate** if and only if there exists a matrix $\sigma \in \mathrm{PGL}_2(k)$ such that $\psi = \sigma^{-1} \phi \sigma$.

Remark 1. *Notice two different uses of notation ϕ^σ for rational functions and f^σ for binary forms.*

The function $\phi^\sigma = \sigma^{-1}\phi\sigma$ is explicitly given as

$$\left(\sigma^{-1}\phi\sigma\right)(x) = \frac{e\, f_0(ax+b,cx+e) - b\, f_1(ax+b,cx+e)}{-c\, f_0(ax+b,cx+e) + a\, f_1(ax+b,cx+e)} = \frac{e\, f_0^\sigma - b\, f_1^\sigma}{-c\, f_0^\sigma + a\, f_1^\sigma} \tag{8}$$

Let $\phi(x,y)$ and $\psi(x,y)$ be degree $d \geq 2$ rational functions given by

$$\phi(x,y) = \frac{f_0(x,y)}{f_1(x,y)} \quad \text{and} \quad \psi(x,y) = \frac{g_0(x,y)}{g_1(x,y)}. \tag{9}$$

By Eq. (8), ϕ and ψ are k-conjugate if and only if there is $\sigma = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \in \mathrm{PGL}_2(k)$ such that

$$g_0 = ef_0^\sigma - bf_1^\sigma \quad \text{and} \quad g_1 = -cf_0^\sigma + af_1^\sigma. \tag{10}$$

Definition 1. For $\phi(x,y) = \frac{f_0(x,y)}{f_1(x,y)}$, define its **associated pair of forms** as

$$\mathcal{I}_\phi := yf_0 - xf_1 \in V_{d+1} \quad \text{and} \quad \mathcal{J}_\phi := \frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y} \in V_{d-1} \tag{11}$$

Lemma 1. *Let $\phi, \psi \in \mathrm{Rat}_d^1$ and $\sigma \in \mathrm{PGL}_2(k)$. Then $\psi = \phi^\sigma$ if and only if*

$$\mathcal{I}_\psi = \mathcal{I}_\phi^\sigma \quad \text{and} \quad \mathcal{J}_\psi = \mathcal{J}_\phi^\sigma.$$

Proof. Let $\phi = \frac{f_0}{f_1}$ and $\psi = \frac{g_0}{g_1}$. Assume that ϕ and ψ are k-conjugate in Rat_d^1, meaning there exists $\sigma \in \mathrm{PGL}_2(k)$ such that $\psi = \phi^\sigma$.

Substituting the expressions for $\mathcal{I}_\psi = yg_0 - xg_1$ and using the values of g_0 and g_1 as in Eq. (10), we have

$$\begin{aligned} \mathcal{I}_\psi &= yg_0 - xg_1 = y(ef_0^\sigma - bf_1^\sigma) - x(-cf_0^\sigma + af_1^\sigma) \\ &= (cx+ey)f_0^\sigma - (ax+by)f_1^\sigma = (yf_0 - xf_1)^\sigma = \mathcal{I}_\phi^\sigma. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{J}_\psi &= \frac{\partial g_0}{\partial x} + \frac{\partial g_1}{\partial y} = \frac{\partial(ef_0^\sigma - bf_1^\sigma)}{\partial x} + \frac{\partial(-cf_0^\sigma + af_1^\sigma)}{\partial y} \\ &= e\frac{\partial f_0^\sigma}{\partial x} - b\frac{\partial f_1^\sigma}{\partial x} - c\frac{\partial f_0^\sigma}{\partial y} + a\frac{\partial f_1^\sigma}{\partial y} = \left(\frac{\partial f_0^\sigma}{\partial x} + \frac{\partial f_1^\sigma}{\partial y}\right)^\sigma = \mathcal{J}_\phi^\sigma. \end{aligned}$$

Thus, we conclude that $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ (and similarly, $\mathcal{J}_\phi$ and $\mathcal{J}_\psi$) are k-equivalent via σ as binary forms.

Conversely, suppose that $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ (respectively, $\mathcal{J}_\phi$ and $\mathcal{J}_\psi$) are k-equivalent via σ as binary forms. This means that $\mathcal{I}_\psi = \mathcal{I}_\phi^\sigma$ and $\mathcal{J}_\psi = \mathcal{J}_\phi^\sigma$. In particular, we have

$$\begin{aligned} \mathcal{I}_\psi &= yg_0 - xg_1 = (yf_0 - xf_1)^\sigma = (cx+ey)f_0^\sigma - (ax+by)f_1^\sigma, \\ \mathcal{J}_\psi &= \frac{\partial g_0}{\partial x} + \frac{\partial g_1}{\partial y} = \frac{\partial f_0^\sigma}{\partial x^\sigma} + \frac{\partial f_1^\sigma}{\partial y^\sigma}. \end{aligned}$$

From the equation for $\mathcal{I}_\psi$, we obtain

$$y\,(g_0 - ef_0^\sigma + bf_1^\sigma) = x\,(g_1 + cf_0^\sigma - af_1^\sigma),$$

which leads to $g_0 - ef_0^\sigma + bf_1^\sigma = x \cdot h(x,y)$, and $g_1 + cf_0^\sigma - af_1^\sigma = y \cdot h(x,y)$, for some $h \in V_{d-1}$. Therefore, we have

$$\begin{aligned}\mathcal{J}_\psi &= 2h + \left(x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y}\right) + e\frac{\partial f_0^\sigma}{\partial x} + b\frac{\partial f_1^\sigma}{\partial x} - c\frac{\partial f_0^\sigma}{\partial y} + a\frac{\partial f_1^\sigma}{\partial y} \\ &= h + \frac{d-1}{2}h + \frac{\partial g_0}{\partial x} + \frac{\partial g_1}{\partial y}.\end{aligned}$$

Thus, we must have $h = 0$, which implies that $\phi^\sigma = \psi$ as claimed. □

Since the pair of binary forms $(\mathcal{I}_\phi, \mathcal{J}_\phi)$ determines the rational function ϕ, we can use the classical theory of binary forms to determine invariants for ϕ. Define

$$\Phi : \mathrm{Rat}_d^1 \to V_{d+1} \oplus V_{d-1}, \tag{12}$$

via $\Phi(\phi) = (\mathcal{I}_\phi, \mathcal{J}_\phi)$. The inverse of Φ is not well-defined since not every pair $(f,g) \in V_{d+1} \oplus V_{d-1}$ determines a rational function.

Definition 2. For any $(f,g) \in V_{d+1} \oplus V_{d-1}$, define the **modular resultant**

$$\Delta_{(f,g)} = \mathrm{Res}\left(xg + \frac{\partial f}{\partial y},\, yg - \frac{\partial f}{\partial x}\right), \tag{13}$$

and the **moduli resultant locus** $\mathcal{N}$ as

$$\mathcal{N} := \{(f,g) \in V_{d+1} \oplus V_{d-1} \mid \Delta_{(f,g)} = 0\}. \tag{14}$$

Then, we have the following result which has also appeared in [19].

Lemma 2. *The map $\Phi : \mathrm{Rat}_d^1 \to (V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$ is bijective. Moreover, for any $(f,g) \in V_{d+1} \oplus V_{d-1} \setminus \mathcal{N}$,*

$$\Phi^{-1}(f,g) = \frac{xg + \frac{\partial f}{\partial y}}{yg - \frac{\partial f}{\partial x}}.$$

Proof. The map Φ is well-defined. Indeed, for $\varphi = f_0/f_1$ one verifies directly that

$$xJ_\varphi + \frac{\partial I_\varphi}{\partial y} = d\, f_0, \qquad yJ_\varphi - \frac{\partial I_\varphi}{\partial x} = d\, f_1,$$

and hence $\Delta_{I_\varphi, J_\varphi} = d^{\,2d}\,\mathrm{Res}(f_0, f_1) \neq 0$, so $\Phi(\varphi) \notin N$.

Let $\phi, \psi \in \mathrm{Rat}_d^1$ such that $\Phi(\phi) = \Phi(\psi)$. Then, $\mathcal{I}_\phi = \mathcal{I}_\psi$ and $\mathcal{J}_\phi = \mathcal{J}_\psi$ as binary forms in V_{d+1} and V_{d-1}, respectively. Since binary forms are defined up to multiplication by a scalar, there exists a diagonal matrix $\sigma \in \mathrm{PGL}_2(k)$ such that $\mathcal{I}_\psi = \mathcal{I}_\phi^\sigma$ and $\mathcal{J}_\psi = \mathcal{J}_\phi^\sigma$. By Lem. 1, we conclude that $\psi = \phi^\sigma = \phi$. Thus, Φ is injective.

Now, assume that $(f,g) \in (V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$. The condition $\Delta_{(f,g)} \neq 0$ ensures that the preimage

$$\Phi^{-1}(f,g) = \frac{xg + \frac{\partial f}{\partial y}}{yg - \frac{\partial f}{\partial x}}$$

belongs to Rat_d^1. Consequently, the map Φ^{-1} is well-defined.

To show that Φ and Φ^{-1} are inverses of each other, we will demonstrate that $\Phi \circ \Phi^{-1} = \mathrm{id}$ on $(V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$ and $\Phi^{-1} \circ \Phi = \mathrm{id}$ on Rat_d^1.

First, compute $\Phi \circ \Phi^{-1}$ on $(V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$: Let $(f,g) \in (V_{d+1} \oplus V_{d-1}) \setminus \mathcal{N}$ and $\phi = \Phi^{-1}(f,g) = \frac{xg + \frac{\partial f}{\partial y}}{yg - \frac{\partial f}{\partial x}}$. Then,

$$\mathcal{I}_\phi = y\left(xg + \frac{\partial f}{\partial y}\right) - x\left(yg - \frac{\partial f}{\partial x}\right) = y\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial x} = (d+1)f \in V_{d+1}$$

$$\mathcal{J}_\phi = \frac{\partial}{\partial x}\left[xg + \frac{\partial f}{\partial y}\right] + \frac{\partial}{\partial y}\left[yg - \frac{\partial f}{\partial x}\right] = 2g + x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 2g + (d-1)g \in V_{d-1}$$

Here $(d+1)f$ and $(d+1)g$ are nonzero scalar multiples of f and g, and thus represent the same binary forms in V_{d+1} and V_{d-1}. Thus, $\mathcal{I}_\phi = f$ and $\mathcal{J}_\phi = g$, so $\Phi(\phi) = (\mathcal{I}_\phi, \mathcal{J}_\phi) = (f,g)$.

Second, compute $\Phi^{-1} \circ \Phi$ on Rat_d^1: Let $\phi = \frac{f_0}{f_1} \in \mathrm{Rat}_d^1$. Then,

$$\begin{aligned}
\left(\Phi^{-1} \circ \Phi\right)(\phi) = \Phi^{-1}(\mathcal{I}_\phi, \mathcal{J}_\phi) &= \Phi^{-1}\left(yf_0 - xf_1, \frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) \\
&= \frac{x\left(\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) + \frac{\partial}{\partial y}\left[yf_0 - xf_1\right]}{y\left(\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial x}\left[yf_0 - xf_1\right]} \\
&= \frac{x\left(\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) + \left(f_0 + y\frac{\partial f_0}{\partial y} - x\frac{\partial f_1}{\partial y}\right)}{y\left(\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial y}\right) - \left(y\frac{\partial f_0}{\partial x} - f_1 - x\frac{\partial f_1}{\partial x}\right)} \\
&= \frac{x\frac{\partial f_0}{\partial x} + y\frac{\partial f_0}{\partial y} + f_0}{x\frac{\partial f_1}{\partial x} + y\frac{\partial f_1}{\partial y} + f_1} = \frac{df_0 + f_0}{df_1 + f_1} = \frac{f_0}{f_1} = \phi.
\end{aligned}$$

Since both compositions $\Phi \circ \Phi^{-1}$ and $\Phi^{-1} \circ \Phi$ are the identity maps, we conclude that Φ and Φ^{-1} are indeed inverse to each other. This completes the proof. □

3.1. **Ring of invariants $\mathcal{R}_{(d+1),(d-1)}$ and a theorem of Clebsch.** The action of $\mathrm{GL}_2(k)$ on V_d induces an action of $\mathrm{GL}_2(k)$ in $V_{d+1} \oplus V_{d-1}$. To determine the isomorphism classes of degree d rational functions, we have to determine the ring of invariants of $V_{d+1} \oplus V_{d-1}$. This is a well known in classical invariant theory. We briefly describe it below; see [19] for details.

Let V be an SL_2-module and $\mathcal{O}(V)$ the algebra of polynomial functions on V. $\mathrm{SL}_2(k)$ acts on $\mathcal{O}(V)$ via

$$M \cdot p(f_1, \ldots, f_r) \to p(M^{-1}f_1, \ldots, M^{-1}f_r),$$

for every $M \in \mathrm{SL}_2(k)$. An invariant of V is an element $\mathcal{T} \in \mathcal{O}(V)$ such that $M\mathcal{T} = \mathcal{T}$, for all $M \in \mathrm{SL}_2(k)$. The set of invariants is denoted by $\mathcal{O}(V)^{\mathrm{SL}_2}$.

A transvectant $(\mathcal{T}, \mathcal{S})_l$ is called **irrelevant** if there exist $\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_1, \mathcal{S}_2$ and l_1, l_2 such that

$$l = l_1 + l_2, \quad \mathcal{T} = \mathcal{T}_1 \cdot \mathcal{T}_2, \quad \mathcal{S} = \mathcal{S}_1 \cdot \mathcal{S}_2,$$

and $l_1 \leq \operatorname{ord}\mathcal{T}_1, \operatorname{ord}\mathcal{S}_1$, and $l_2 \leq \operatorname{ord}\mathcal{T}_2, \operatorname{ord}\mathcal{S}_2$. A transvectant which is not irrelevant is called **relevant**.

Let V and W be two SL_2-modules whose covariants are finitely generated, and assume

$$\begin{aligned}
&\mathcal{T}_1, \ldots, \mathcal{T}_r: \quad \text{are the generators of the covariants of } V \\
&\mathcal{S}_1, \ldots, \mathcal{S}_s: \quad \text{are the generators of the covariants of } W.
\end{aligned} \tag{15}$$

Theorem (Clebsch)**.** *The ring of covariants of $V \oplus W$ is finitely generated. Moreover, a finite generating system can be chosen from the set of all transvectants*

$$(\mathcal{T}, \mathcal{S})_l\,, \quad \textit{for} \quad l \geq 0,$$

where $\mathcal{T}$ is a monomial in the $\mathcal{T}_i$'s and $\mathcal{S}$ a monomial in the $\mathcal{S}_j$'s. In other words, by the relevant transvectants $(\mathcal{T}, \mathcal{S})_l$.

3.2. $\operatorname{Proj} \mathcal{R}_{(d+1,d-1)}$ **as a weighted projective space.** Let $\xi_0, \ldots, \xi_n$ be a generating system of $\mathcal{R}_{(d+1,d-1)}$. Since all $\xi_0, \ldots, \xi_n$ are homogeneous polynomials, then $\mathcal{R}_{(d+1,d-1)}$ is a graded ring and $\operatorname{Proj} \mathcal{R}_{(d+1,d-1)}$ is a weighted projective space.

Let $\omega := (q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}$ be a fixed tuple of positive integers called **weights**. Consider the action of $k^\times = k \setminus \{0\}$ on $\mathbb{A}^{n+1}(k)$ as follows:

$$\lambda \star (x_0, \ldots, x_n) = (\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n)$$

for $\lambda \in k^\times$. The quotient of this action is called a **weighted projective space** and denoted by $\mathbb{P}^n_\omega(k)$. It is the projective variety $\operatorname{Proj}(k[x_0, \ldots, x_n])$ associated to the graded ring $k[x_0, \ldots, x_n]$ where the variable x_i has degree q_i for $i = 0, \ldots, n$.

We will denote a point $\mathfrak{p} \in \mathbb{P}^n_\omega(k)$ by $\mathfrak{p} = [x_0 : x_1 : \cdots : x_n]$. Let $\phi(x, y) \in \operatorname{Rat}^1_d$ given by $\phi(x, y) = [f_0 : f_1]$. Its associated binary forms $\mathcal{I}_\phi \in V_{d+1}$ and $\mathcal{J}_\phi \in V_{d-1}$, and $\xi_0, \xi_1, \ldots, \xi_n$ the generators of the ring of invariants $\mathcal{R}_{(d+1,d-1)}$.

The **invariants** of the rational function ϕ are defined as

$$\xi(\phi) := [\xi_0(\mathcal{I}_\phi, \mathcal{J}_\phi), \xi_1(\mathcal{I}_\phi, \mathcal{J}_\phi), \ldots, \xi_n(\mathcal{I}_\phi, \mathcal{J}_\phi)] \in \mathbb{P}^n_\omega(k). \tag{16}$$

Moreover, $\phi = \psi^\sigma$ for $\sigma \in \mathrm{GL}_2(k)$ if and only if $\xi(\phi) = \lambda \star \xi(\psi)$, for $\lambda = (\det \sigma)^{\frac{d}{2}}$.

Next we will determine explicitly invariants of $\mathcal{R}_{d+1,d-1}$. From now on $f \in V_{d+1}$ and $g \in V_{d-1}$ where

$$f = \sum_{i=0}^{d+1} a_i x^i y^{d+1-i} \quad \text{and} \quad g = \sum_{i=0}^{d-1} b_i x^i y^{d-1-i}. \tag{17}$$

3.3. **Invariants of $V_4 \oplus V_2$.** To apply this framework, we specialize to $d = 3$, where $\mathcal{I}_\phi \in V_4$ and $\mathcal{J}_\phi \in V_2$. We take $d = 3$, $f \in V_4$ and $g \in V_2$ as in Eq. (17)

$$\begin{aligned} f(x, y) &= a_4 x^4 + a_3 x^3 y + a_2 x^2 y^2 + a_1 x y^3 + a_0 y^4 \\ g(x, y) &= b_2 x^2 + b_1 x y + b_0 y^2 \end{aligned} \tag{18}$$

The generators of covariants of V_4 and V_2 are

$$T = \{f,\, \mathcal{T} = (f, f)_2,\, \mathcal{T}_2 = (f, f)_4,\, \mathcal{T}_3 = ((f, f)_2, f)_4\} \quad \text{and} \quad S = \{g,\, \mathcal{S}_2 = (g, g)_2\}$$

respectively. Hence, we are considering all transvectants

$$(f^{m_1} \mathcal{T}^{m_2} \mathcal{T}_2^{m_3} \mathcal{T}_3^{m_4}, g^{s_1} S_2^{s_2})_l\,,$$

for some m_1, m_2, m_3, m_4, s_1, s_2. Since $\mathcal{T}_2, \mathcal{T}_3$ and S_2 are invariants, their exponents must be zero, otherwise we get reducible invariants. Hence, $\mathcal{T}_2, \mathcal{T}_3, S_2$ are part of the generating set and further we only consider $(f^{m_1} \mathcal{T}^{m_2}, g^s)_l$. Then the relevant transvectants are $S_2, \mathcal{T}_2, \mathcal{T}_3$, and $R_3 := (\mathcal{T}, g^2)_4$, $R_4 := (f, g^2)_4$, $R_6 := (g^3, (f, \mathcal{T})_1)_6$. Hence, the set of invariants is $\xi(\phi) = (\xi_0, \ldots, \xi_5)$, where

$$\xi_0 = (g, g)_2,\ \xi_1 = (f, f)_4,\ \xi_2 = (\mathcal{T}, f)_4,\ \xi_3 = R_3,\ \xi_4 = R_4,\ \xi_5 = R_6$$

with weights $(2,2,3,3,4,6)$ respectively. Let $d=3$ and f,g as in Eq. (17). We have the following expressions for invariants:

$$\begin{aligned}
\xi_0 &= \frac{1}{2}\left(4b_0b_2 - b_1^2\right)\\
\xi_1 &= \frac{1}{6}\left(a_2^2 - 3a_1a_3 + 12a_0a_4\right)\\
\xi_2 &= \frac{1}{72}\left(-2a_2^3 + 9(a_1a_3 + 8a_0a_4)a_2 - 27\left(a_4a_1^2 + a_0a_3^2\right)\right)\\
\xi_3 &= \frac{1}{6}\left(6a_4b_0^2 - 3a_3b_1b_0 + 2a_2b_2b_0 + a_2b_1^2 + 6a_0b_2^2 - 3a_1b_1b_2\right)\\
\xi_4 &= -\frac{1}{72}\left(2a_2^2b_1^2 + 4a_2^2b_0b_2 - 24a_4a_2b_0^2 - 24a_0a_2b_2^2 - 6a_3a_2b_0b_1\right.\\
&\quad -6a_1a_2b_1b_2 + 9a_3^2b_0^2 - 3a_1a_3b_1^2 - 24a_0a_4b_1^2 + 9a_1^2b_2^2 + 36a_1a_4b_0b_1 - 6a_1a_3b_0b_2\\
&\quad \left.-48a_0a_4b_0b_2 + 36a_0a_3b_1b_2\right)\\
\xi_5 &= -\frac{1}{32}\left(a_3^3b_0^3 + 8a_1a_4^2b_0^3 - 4a_2a_3a_4b_0^3 - a_2a_3^2b_1b_0^2 - 8a_0a_1a_4b_1^2b_2 - 16a_0a_4^2b_1b_0^2\right.\\
&\quad -2a_1a_3a_4b_1b_0^2 + a_1a_3^2b_2b_0^2 - 4a_1a_2a_4b_2b_0^2 + 8a_0a_3a_4b_2b_0^2 + a_1a_3^2b_1^2b_0 + 4a_2^2a_4b_1b_0^2\\
&\quad -4a_1a_2a_4b_1^2b_0 + 8a_0a_3a_4b_1^2b_0 - a_1^2a_3b_2^2b_0 + 4a_0a_2a_3b_2^2b_0 - 8a_0a_1a_4b_2^2b_0\\
&\quad -6a_0a_3^2b_1b_2b_0 + 6a_1^2a_4b_1b_2b_0 - a_0a_3^2b_1^3 + a_1^2a_4b_1^3 - a_1^3b_2^3 + 4a_0a_1a_2b_2^3 - 8a_0^2a_3b_2^3\\
&\quad \left.-4a_0a_2^2b_1b_2^2 + a_1^2a_2b_1b_2^2 + 2a_0a_1a_3b_1b_2^2 + 16a_0^2a_4b_1b_2^2 - a_1^2a_3b_1^2b_2 + 4a_0a_2a_3b_1^2b_2\right)
\end{aligned}$$

The ring of invariants $\mathcal{R}_{4,2}$ is generated by $\xi_0,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5$ and a relation between invariants $\xi_0,\ldots,\xi_5$ is, according to [19], satisfy the following equation

$$\xi_5^2 = \frac{1}{108}\xi_0^3\xi_1^3 - 18\xi_0^3\xi_2^2 - \frac{1}{24}\xi_0\xi_1^2\xi_3^2 - \frac{1}{6}\xi_2\xi_3^3 + \frac{1}{2}\xi_0\xi_2\xi_3\xi_4 + \frac{1}{4}\xi_1\xi_3^2\xi_4 - \frac{1}{4}\xi_0\xi_1\xi_4^2 - \frac{1}{2}\xi_4^3 \tag{19}$$

Given a rational cubic

$$\phi(x) = \frac{f_0(x)}{f_1(x)} = \frac{c_0x^3 + c_1x^2 + c_2x + c_3}{c_4x^3 + c_5x^2 + c_6x + c_7} \tag{20}$$

we compute its invariants in terms of its coefficients. We have $I_6 = \mathrm{Res}(f_0,f_1)$.

$$\begin{aligned}
I_6 =& c_3^3c_4^3 - c_0^3c_7^3 + c_3c_0^2c_6^3 - c_2^3c_7c_4^2 + c_1^3c_7^2c_4 - c_3^2c_2c_5c_4^2 - c_3^2c_0c_5^3 - 2c_2^2c_1c_6c_4^2\\
&+ c_3c_2^2c_6c_4^2 + c_3^2c_1c_5^2c_4 - 3c_3^2c_0c_7c_4^2 - c_1^2c_0c_7^2c_5 + c_1c_0^2c_7^2c_6 + 2c_2c_0^2c_7^2c_5\\
&+ 3c_3c_0^2c_7^2c_4 - c_2^2c_0c_7c_5^2 + 3c_3^2c_0c_6c_5c_4 - 2c_3c_2c_0c_6^2c_4 + c_3c_2c_0c_6c_5^2\\
&+ 3c_3c_2c_1c_7c_4^2 + 2c_3c_1c_0c_7c_5^2 - c_3c_1c_0c_6^2c_5 - c_2c_1^2c_7c_6c_4 - 3c_2c_1c_0c_7^2c_4\\
&+ c_2^2c_1c_7c_5c_4 + 2c_2^2c_0c_7c_6c_4 - c_3c_2c_1c_6c_5c_4 - c_3c_2c_0c_7c_5c_4 + c_3c_1c_0c_7c_6c_4\\
&+ c_3c_1^2c_6^2c_4 - c_2c_0^2c_7c_6^2 + c_2c_1c_0c_7c_6c_5 - 3c_3c_0^2c_7c_6c_5 - 2c_3c_1^2c_7c_5c_4
\end{aligned}$$

The pair of binary forms $\mathcal{I}_\phi$ and $\mathcal{J}_\phi$ associated to ϕ are

$$\begin{aligned}
\mathcal{I}_\phi &= c_3x^3y + c_2x^2y^2 + c_1xy^3 + c_0y^4 - c_7x^4 - c_6x^3y - c_5x^2y^2 - c_4xy^3\\
\mathcal{J}_\phi &= 3c_3x^2 + 2c_2xy + c_1y^2 + c_6x^2 + 2c_5xy + 3c_4y^2
\end{aligned} \tag{21}$$

Next we evaluate the following transvectants

$$\begin{aligned}
&\xi_0 = (\mathcal{J}_\phi,\mathcal{J}_\phi)_2\,, \quad \xi_1 = (\mathcal{I}_\phi,\mathcal{I}_\phi)_4\,, && \xi_2 = ((\mathcal{I}_\phi,\mathcal{I}_\phi)_2,\mathcal{I}_\phi)_4\,,\\
&\xi_3 = \left(\mathcal{I}_\phi,\mathcal{J}_\phi^2\right)_4\,, \quad \xi_4 = \left((\mathcal{I}_\phi,\mathcal{I}_\phi)_2,\mathcal{J}_\phi^2\right)_4\,, && \xi_5 = \left(\mathcal{J}_\phi^3,(\mathcal{I}_\phi,(\mathcal{I}_\phi,\mathcal{I}_\phi)_2)_1\right)_6
\end{aligned} \tag{22}$$

Remark 2. *The full symbolic expressions for the invariants $\xi_0, \ldots, \xi_5$ and the auxiliary identities used in their computation follow directly from the formulas given above. We display them in* [1].

The modular resultant $\Delta_{\mathcal{I}_\phi, \mathcal{J}_\phi} = \mathrm{Res}(\mathcal{I}_\phi, \mathcal{J}_\phi)$ is a homogeneous polynomial of degree six in terms of $c_0, \ldots, c_7$. Hence, we will denote it by $J_6 := \mathrm{Res}(\mathcal{I}_\phi, \mathcal{J}_\phi)$.

Thus, there are three invariants of degree 6 in the case of cubics, namely I_6, J_6, and ξ_5. We would like to express I_6 and J_6 in terms of $\xi_0, \ldots, \xi_5$. The expression of I_6 was computed in [19, pg. 38]

$$I_6 = \frac{1}{8}\xi_1^3 + \frac{1}{384}\xi_1\xi_0^2 - \frac{3}{4}\xi_2^2 - \frac{3}{16}\xi_2\xi_4 + \frac{1}{256}\xi_4^2 + \frac{3}{16}\xi_1\xi_3 - \frac{1}{64}\xi_0\xi_3 - \frac{1}{8}\xi_5 \tag{23}$$

It seems there are a couple of typos in the printed version of [19, pg. 38], and we could not verify it directly. It can be easily noticeable that it is incorrect since it is not a homogeneous polynomial of degree 6; see for example the monomials ξ_4^2, ξ_1, ξ_3, $\xi_0\xi_3$ which are not of degree 6.

The correct expression can be derived using computational algebra. This involves expressing I_6 and J_6 as linear combinations of the 10 degree 6 invariants: ξ_5, $\xi_4\xi_0$, $\xi_4\xi_1$, $\xi_2^{2-j}\xi_3^j$, $\xi_0^{3-i}\xi_1^i$, for $i = 0, 1, 2, 3$ and $j = 0, 1, 2$. We have

$$I_6 = -\frac{1}{8}\xi_1^3 - \frac{1}{384}\xi_0^2\xi_1 + \frac{3}{4}\xi_2^2 - \frac{3}{16}\xi_1\xi_4 - \frac{1}{256}\xi_3^2 + \frac{3}{16}\xi_2\xi_3 + \frac{1}{64}\xi_0\xi_4 - \frac{1}{8}\xi_5$$

$$J_6 = \xi_3^2 - 4\xi_4\xi_0 + \frac{2}{3}\xi_0^2\xi_1$$

3.4. **A natural map from weighted projective space to projective space.** Let $\mathbb{P}(q_0, \ldots, q_n) = \mathrm{Proj}\, k[x_0, \ldots, x_n]$, $\deg(x_i) = q_i$, and $d = \mathrm{lcm}(q_0, \ldots, q_n)$. Then the monomials x_0^{d/q_0}, x_1^{d/q_1}, $\ldots$, x_n^{d/q_n} all have weighted degree d. Hence they define a morphism $\phi_d : \mathbb{P}(q_0, \ldots, q_n) \longrightarrow \mathbb{P}^n$ given by

$$[x_0 : \cdots : x_n] \longmapsto [x_0^{d/q_0} : \cdots : x_n^{d/q_n}].$$

The following is a well known fact on weighted projective spaces. We include its proof due to lack of a precise reference.

Lemma 3. *The morphism $\phi_d : \mathbb{P}(q_0, \ldots, q_n) \to \mathbb{P}^n$ is finite. The locus where ϕ_d fails to be injective is exactly the union of coordinate strata $x_{i_1} = \cdots = x_{i_r} = 0$ for which* $\gcd(q_j : j \notin \{i_1, \ldots, i_r\}) > 1$.

Proof. The map ϕ_d is induced by the $\mathbf{G}_m$-equivariant morphism

$$F : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{A}^{n+1} \setminus \{0\}, \qquad (x_0, \ldots, x_n) \longmapsto (x_0^{d/q_0}, \ldots, x_n^{d/q_n}),$$

since each coordinate has weight d. Suppose two points represent the same image under ϕ_d. Then there exists $t \in k^*$ such that

$$x_i'^{\,d/q_i} = t\, x_i^{d/q_i} \qquad (i = 0, \ldots, n).$$

Thus $x_i' = \zeta_i\, t^{q_i/d} x_i$, where ζ_i is a (d/q_i)-th root of unity. For the points to represent the same point of $\mathbb{P}(q_0, \ldots, q_n)$ there must exist $\lambda \in k^*$ such that $x_i' = \lambda^{q_i} x_i$. Comparing the two expressions shows that the ambiguity is exactly given by the roots of unity μ_g, where $g = \gcd(q_i : x_i \neq 0)$. Therefore the map is injective precisely when $g = 1$, and it fails to be injective exactly along the coordinate strata where the weights of the nonzero coordinates have a common divisor greater than one. □

Now we turn our attention to our space $\mathbb{P}_{(2,2,3,3,4,6)}$. We consider the Veronese-type morphism

$$[\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, I_6] \to [\xi_0^6, \xi_1^6, \xi_2^4, \xi_3^4, \xi_4^3, I_6^2] \tag{24}$$

Since $I_6 \neq 0$, we can divide by I_6^2 and represent each point as

$$\left[\frac{\xi_0^6}{I_6^2} : \frac{\xi_1^6}{I_6^2} : \frac{\xi_2^4}{I_6^2} : \frac{\xi_3^4}{I_6^2} : \frac{\xi_4^3}{I_6^2} : 1\right] \in \mathbb{P}^5.$$

The map fails to be injective along the strata where the weights of the nonzero coordinates have a common divisor (i.e., where the weighted $\mathbf{G}_m$-action has nontrivial stabilizer). This motivates the definition of the following invariants

$$i_1 = \frac{\xi_0^6}{I_6^2},\ i_2 = \frac{\xi_1^6}{I_6^2},\ i_3 = \frac{\xi_2^4}{I_6^2},\ i_4 = \frac{\xi_3^4}{I_6^2},\ i_5 = \frac{\xi_4^3}{I_6^2}, \tag{25}$$

which are $\mathrm{GL}_2(k)$-invariants and are defined everywhere in the moduli space Rat_3^1. We call such invariants $i_1, \ldots, i_5$ **absolute invariants** of $\phi(x, y)$. Hence, we have:

Proposition 1. *Two degree-three rational functions are conjugate if and only if they have the same absolute invariants, provided that the gcd of the weights of the non-zero coordinates among $\{\xi_0(\phi), \ldots, \xi_5(\phi)\}$ equals 1.*

Proof. Let $\phi, \psi \in \mathrm{Rat}_3^1$ be given by $\phi = \frac{f_0}{f_1}$, $\psi = \frac{g_0}{g_1}$, and let $\xi(\phi)$ and $\xi(\psi)$ be their weighted projective invariants as in Eq. (16), with absolute invariants as in Eq. (25). Let $S \subseteq \{0, 1, 2, 3, 4, 5\}$ be the common support of the non-zero coordinates among $\{\xi_0(\phi), \ldots, \xi_5(\phi)\}$ By assumption $\gcd(w_i : i \in S) = 1$.

If $\psi = \phi^\sigma$ for some $\sigma \in \mathrm{PGL}_2(k)$, then by Lem. 1 we have $\mathcal{I}_\psi = \mathcal{I}_\phi^\sigma$ and $\mathcal{J}_\psi = \mathcal{J}_\phi^\sigma$, and since the ξ_i are $\mathrm{SL}_2(k)$–invariants,

$$\xi_i(\psi) = \xi_i(\mathcal{I}_\psi, \mathcal{J}_\psi) = \xi_i(\mathcal{I}_\phi^\sigma, \mathcal{J}_\phi^\sigma) = \xi_i(\phi).$$

If $\sigma \in \mathrm{GL}_2(k)$ with $\det \sigma = \lambda$, then $I_6(\phi^\sigma) = \lambda^3 I_6(\phi)$ and $\xi_i(\phi^\sigma) = \lambda^{w_i/2}\, \xi_i(\phi)$, where $(w_0, \ldots, w_5) = (2, 2, 3, 3, 4, 6)$. Hence

$$i_j(\psi) = \frac{\xi_{j-1}(\psi)^{12/w_{j-1}}}{I_6(\psi)^2} = \frac{(\lambda^{w_{j-1}/2}\xi_{j-1}(\phi))^{12/w_{j-1}}}{(\lambda^3 I_6(\phi))^2} = i_j(\phi).$$

Thus conjugate maps have the same absolute invariants.

Now suppose $i_j(\phi) = i_j(\psi)$ for all $j = 1, \ldots, 5$. For each j we have $\frac{\xi_{j-1}(\phi)^{12/w_{j-1}}}{I_6(\phi)^2} = \frac{\xi_{j-1}(\psi)^{12/w_{j-1}}}{I_6(\psi)^2}$, hence

$$\left(\frac{\xi_{j-1}(\phi)}{\xi_{j-1}(\psi)}\right)^{12/w_{j-1}} = \left(\frac{I_6(\phi)}{I_6(\psi)}\right)^2. \tag{26}$$

Set $\rho_j = \frac{\xi_{j-1}(\phi)}{\xi_{j-1}(\psi)}$ and $c = \left(\frac{I_6(\phi)}{I_6(\psi)}\right)^2$. Since k is algebraically closed with $\mathrm{char}(k) = 0$, choose $\lambda \in k^\times$ with $\lambda^{12} = c$. Then every solution of Eq. (26) has the form

$$\rho_j = \lambda^{w_{j-1}}\zeta_j, \qquad \zeta_j^{12/w_{j-1}} = 1, \tag{27}$$

because $12/w_{j-1}$ is integral for all weights $w_{j-1} \in \{2, 2, 3, 3, 4, 6\}$. The absolute invariants $i_1, \ldots, i_5$ are weight–zero rational functions of the weighted coordinates

$\xi_0, \ldots, \xi_5$. Every monomial occurring in their numerators and denominators has weighted degree

$$\sum_{i=0}^{5} a_i w_i = 0. \tag{28}$$

Substituting the expressions Eq. (27) into such a monomial (with support in S) yields the factor $\prod_{i\in S} \zeta_i^{a_i}$. Since $i_j(\phi) = i_j(\psi)$ for every j, all such monomials must satisfy

$$\prod_{i\in S} \zeta_i^{a_i} = 1 \tag{29}$$

for every exponent vector $(a_i)_{i\in S}$ satisfying the degree–zero condition Eq. (28).

The vectors $(a_i)_{i\in S}$ arising from the weight–zero monomials span the kernel of the restricted weight map

$$\mathbb{Z}^S \longrightarrow \mathbb{Z}, \qquad (a_i)_{i\in S} \longmapsto \sum_{i\in S} a_i w_i.$$

Because $\gcd(w_i : i \in S) = 1$, this kernel contains no nontrivial torsion. Therefore the only solution of Eq. (29) is $\zeta_i = 1$ for all $i \in S$. Hence $\rho_j = \lambda^{w_{j-1}}$ for all j, i.e. $\xi_{j-1}(\phi) = \lambda^{w_{j-1}} \xi_{j-1}(\psi)$. Thus $\xi(\phi) = \lambda \star \xi(\psi)$, so ϕ and ψ represent the same point of the weighted projective space $\mathbb{P}(2, 2, 3, 3, 4, 6)$ and therefore the same conjugacy class in $\mathrm{PGL}_2(k)$. □

We now define the map $\Phi : \mathbb{C}^8 \setminus \{I_6 = 0\} \longrightarrow \mathbb{C}^5$ by

$$(c_0, \ldots, c_7) \to (i_1(c_0, \ldots, c_7), \ldots, i_5(c_0, \ldots, c_7)), \tag{30}$$

where $i_1, \ldots, i_5$ are the absolute invariants defined in Eq. (25). The condition $I_6 \neq 0$ ensures that these invariants are defined everywhere on the moduli space.

To study the image $\mathrm{Img}(\Phi)$, we consider the Jacobian matrix $J(\Phi)$, a 5×8 matrix with entries

$$\frac{\partial i_j}{\partial c_k}, \qquad j = 1, \ldots, 5,\ k = 0, \ldots, 7.$$

Since the rank of $J(\Phi)$ is at most 5, it governs the local geometry of the map Φ. If $\mathrm{rank}(J(\Phi)) = 5$, then Φ is a submersion at the corresponding point, and by the Implicit Function Theorem the image $\mathrm{Img}(\Phi)$ is locally a smooth 5–dimensional manifold near $\Phi(c_0, \ldots, c_7)$.

If $\mathrm{rank}(J(\Phi)) < 5$, that is, if all 5×5 minors vanish, then Φ fails to be a submersion and the image may develop singularities. Geometrically, these points correspond to rational functions possessing nontrivial automorphisms. Equivalently, they are precisely the loci where the map described in Prop. 1 fails to be injective, i.e., where the gcd condition of Prop. 1 is violated.

Thus the degeneracy of the Jacobian reflects the presence of additional symmetry. This suggests that the singular locus of $\mathrm{Img}(\Phi)$, characterized by the condition $\mathrm{rank}(J(\Phi)) < 5$, coincides with the union of the automorphism strata. In particular, we expect it to align with the locus $\{J_6 = 0\} \subset \mathbb{C}^8$. While a direct computational verification of this statement is difficult, the geometric picture is clear: the points where Prop. 1 fails correspond exactly to maps with extra automorphisms. These loci form the automorphism strata $\mathcal{L}_1, \ldots, \mathcal{L}_7$, whose structure and explicit parametrizations are described in the next section.

4. Automorphisms

In this section we will define and study the automorphism groups of rational functions. Such groups have long been the focus of many authors in arithmetic dynamics; see [5, 7, 10, 19]. The approach here draws on the analogy with the automorphism groups of hyperelliptic curves; see [13, 14]. First we recall some preliminaries.

Let G be a finite subgroup of $\mathrm{PGL}_2(k)$. Therefore G is isomorphic to one of the following: C_n, D_n, A_4, S_4, or A_5. We can represent their generators as matrices:

(31)

$$i)\ C_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & 1 \end{bmatrix} \right\rangle,\ ii)\ D_n \cong \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \zeta_n & 0 \\ 0 & 1 \end{bmatrix} \right\rangle,\ iii)\ A_4 \cong \left\langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \zeta_4 \\ 1 & -\zeta_4 \end{bmatrix} \right\rangle$$

$$iv)\quad S_4 \cong \left\langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right\rangle,\ v)\ A_5 \cong \left\langle \begin{bmatrix} \omega & 1 \\ 1 & -\omega \end{bmatrix}, \begin{bmatrix} \omega & \zeta_5^4 \\ 1 & -\zeta_5^4\omega \end{bmatrix} \right\rangle$$

where $\omega = \frac{-1+\sqrt{5}}{2}$, ζ_n is a primitive n^{th} root of unity and $i = \sqrt{-1}$.

Remark 3. *In each case above, there is $\sigma \in G$ which fixes 0 and ∞. The proof is elementary. In the first two cases, the Möbius transformation $\sigma(x) = \zeta_m x$ will fix 0 and ∞. In the next two cases, $\sigma(x) = -x$ will do that, and in the last case, $\sigma(x) = \omega x$ is in the group and will fix 0 and ∞. This property constrains $\mathrm{Aut}(\phi)$ later.*

Let $\phi \in \mathrm{Rat}_d^1$. A point $[x_0 : x_1] \in \mathbb{P}^1$ is called a **fixed point** for ϕ if $\phi(x_0, x_1) = (x_0, x_1)$. Let $t = x_1/x_0$. Hence, ϕ can be taken as a rational function in t, say $\phi(t) = \frac{F(t)}{G(t)}$. Then t is a fixed point if $\phi(t) = t$, which implies that

$$S(t) := F(t) - t\,G(t) = 0$$

which is at most a degree $(d+1)$ equation in t. Hence, a degree d rational function has at most $(d+1)$ fixed points.

We denote the set of fixed points of ϕ by $\mathbf{Fix}(\phi)$. Notice that if $\mathbf{Fix}(\phi) = \{w_1, \ldots, w_s\}$ with $s \leq d+1$ is known, then we can uniquely determine the rational function ϕ by solving the linear system $F(w_i) - w_i\,G(w_i) = 0$, for $i = 1, \ldots, s$, assuming $s = d+1$. A function ϕ has less than $d+1$ fixed points exactly when the discriminant $\Delta(S, t) = 0$.

An **automorphism** of ϕ is called a $\sigma \in \mathrm{PGL}_2(k)$ such that $\phi \circ \sigma = \sigma \circ \phi$. The set of automorphisms of ϕ is denoted by

$$\mathrm{Aut}(\phi) := \{\sigma \in \mathrm{PGL}_2(k) : \phi^\sigma = \phi\}.$$

It forms a group. For any $\sigma \in \mathrm{Aut}(\phi)$, by $\mathbf{Fix}(\sigma)$ we denote its set of fixed points.

Remark 4. *As in the case of curves, there is some confusion in the literature on what is called the **automorphism group** of ϕ. Throughout this paper, by $G := \mathrm{Aut}(\phi)$, we mean the **full automorphism group** of $\phi(x)$ over the algebraic closure of k and not simply some $G \hookrightarrow \mathrm{Aut}(\phi)$ as it is used frequently by many authors.*

For an automorphism $\sigma \in \mathrm{PGL}_2(k)$ of order s, we denote by $\mathcal{L}_d^\sigma$ the locus of degree-d rational maps fixed by σ,

$$\mathcal{L}_d^\sigma = \{\, \varphi \in \mathrm{Rat}_d^1 : \varphi^\sigma = \varphi \,\}.$$

Now we have the following:

Lemma 4. *Let $\phi \in \mathrm{Rat}_d^1$ and $\sigma \in \mathrm{Aut}(\phi)$. Then*

(i) *If* $p \in \mathbf{Fix}(\sigma)$, *then* $\phi(p) \in \mathbf{Fix}(\sigma)$.
(ii) *If* $w \in \mathbf{Fix}(\phi)$, *then* $\sigma(w) \in \mathbf{Fix}(\phi)$.

Hence, $\mathrm{Aut}(\phi)$ *acts on* $\mathbf{Fix}(\phi)$ *by permutation. Moreover, if* $\sigma \in \mathrm{Aut}(\phi)$ *is an automorphism of order* m, *then* m *divides the cardinality of* $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$. *Then* $\dim \mathcal{L}_d^\sigma = \delta = s - 1$, *where* s *is the number of orbits on* $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$.

Proof. For any $p \in \mathbf{Fix}(\phi)$, $\sigma(p) \in \mathbf{Fix}(\phi)$ since

$$\phi(\sigma(p)) = \sigma(\phi(p)) = \sigma(p),$$

which implies that $\sigma(p) \in \mathbf{Fix}(\phi)$. If $w \in \mathbf{Fix}(\phi)$ then

$$\sigma(w) = \sigma(\phi(w)) = \phi(\sigma(w)),$$

which implies that $\sigma(w) \in \mathbf{Fix}(\phi)$.

Since $\langle\sigma\rangle$ has no fixed points in $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$, then it acts transitively on $\mathbf{Fix}(\phi) \setminus \mathbf{Fix}(\sigma)$. Hence, $|\sigma|$ divides its cardinality.

Since any automorphism $\sigma \in \mathrm{PGL}_2(k)$ of order s is conjugate to one fixing 0 and ∞, and the relation $\varphi^\sigma = \varphi$ is preserved under simultaneous conjugation of φ and σ, we may assume without loss of generality that σ fixes 0 and ∞. Hence, the dimension of the family of rational functions $\phi(x)$ is one less than the number of roots of F and G. Hence, this number is exactly $s - 1$. □

Proposition 2. *Let* $\sigma \in \mathrm{Aut}(\phi)$ *such that* $|\sigma| = m$. *Then* $H := \langle\sigma\rangle$ *acts on* $\phi^{-1}(0)$ *and* $\phi^{-1}(\infty)$. *Hence,* $\phi(x)$ *can be written as*

$$\phi(x) = x\,\psi(x^m) \quad \text{or} \quad \phi(x) = \frac{1}{x}\,\psi(x^m),$$

where $\psi(x)$ *is a rational function. Moreover, for* $G \cong A_4, S_4, A_5$ *then* $m = 2, 4, 5$.

Proof. Let $\sigma \in G$ and $t \in \mathbf{Fix}(\sigma)$. For each $\alpha \in \phi^{-1}(t)$ we have

$$\phi(\sigma(\alpha)) = \sigma(\phi(\alpha)) = \sigma(t) = t.$$

Then $\langle\sigma\rangle$ acts on the fiber $\phi^{-1}(t)$. From Remark 3 there is $\sigma \in G$ which fixes 0 and ∞. Then $\langle\sigma\rangle$ acts on $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$. Then points in $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$ are

$$\alpha_1, \xi\alpha_1, \ldots, \xi^{m-1}\alpha_1,\ \alpha_2, \xi\alpha_2, \ldots, \xi^{m-1}\alpha_2,\ \ldots, \alpha_r, \xi\alpha_r, \ldots, \xi^{m-1}\alpha_r,$$
$$\beta_1, \xi\beta_1, \ldots, \xi^{m-1}\beta_1,\ \beta_2, \xi\beta_2, \ldots, \xi^{m-1}\beta_2,\ \ldots, \beta_r, \xi\beta_r, \ldots, \xi^{m-1}\beta_r,$$

where $r = \frac{d-1}{m}$ and $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r \in k \setminus \{0, 1, \infty\}$. Therefore,

$$\mathbf{F}(x) = \prod_{j=1}^{r} \prod_{i=0}^{m-1} \left(x - \xi_m^i \alpha_j\right) = \prod_{j=1}^{r} (x^m - \alpha_j^m),$$
$$\mathbf{G}(x) = \prod_{j=1}^{r} \prod_{i=0}^{m-1} \left(x - \xi_m^i \beta_j\right) = \prod_{j=1}^{r} (x^m - \beta_j^m)$$

Thus, $\phi(x)$ can be written as

$$\phi(x) = x\,\frac{\mathbf{F}(x^m)}{\mathbf{G}(x^m)} = x\,\frac{x^{rm} + a_{r-1}x^{(r-1)m} + \cdots a_1 x^m + a_0}{x^{rm} + b_{r-1}x^{(r-1)m} + \cdots b_1 x^m + b_0}$$

or $\phi(x) = \frac{\mathbf{G}(x^m)}{x\,\mathbf{F}(x^m)}$. This completes the proof. □

The above Lemmas give us a way to determine $\phi(x)$ once an automorphism $\sigma \in \mathrm{Aut}(\phi)$ is given.

4.1. **Automorphism Groups of Rational Cubics.** The automorphism groups for cubic rational functions and their parametric families were determined in [7] and in [19]. Here we give a brief treatment to sort out some mistakes and furthermore determine explicitly families in terms of invariants $\xi_0, \ldots, \xi_5$ and in each case give a formula for the rational cubic defined over its field of moduli.

Lemma 5. *Let $\phi \in \mathrm{Rat}_d^1$ with $d = 3$. Then the following hold:*

(i) *Elements of $\mathrm{Aut}(\phi)$ have orders at most 4.*

(ii) *$\mathrm{Aut}(\phi)$ is isomorphic to one of the following: $\{e\}, C_2, C_3, C_4, V_4, D_4$, or A_4.*

Proof. Suppose that $\sigma \in \mathrm{Aut}(\phi)$ is of order n. There is no loss of generality in assuming

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & \zeta_n \end{bmatrix}, \qquad \zeta_n \text{ a primitive } n\text{-th root of unity.}$$

Then $\phi^\sigma = \phi$ if and only if ϕ commutes with σ. Since $\varphi = F/G$ has degree 3, the monomials $x^{r_0}y^{3-r_0}$ in F and $x^{r_1}y^{3-r_1}$ in G must transform by the same character under $\sigma(x) = \zeta_n x$. This forces

$$\zeta_n^{r_0 - r_1 - 1} = 1, \qquad \text{i.e. } n \mid (r_0 - r_1 - 1).$$

Thus only those pairs of monomials with $n \mid (r_0 - r_1 - 1)$ can appear simultaneously in (F, G).

As a result, we only need to characterize cubic rational functions that have an automorphism of order 2, 3, or 4 in terms of their invariants (recall that if $\sigma \in \mathrm{Aut}(\phi)$ has order 4, then $\sigma^2 \in \mathrm{Aut}(\phi)$ has order 2). From (i), we eliminate C_n and D_{2n} for $n \geq 5$, as well as A_5, from consideration. Thus, possible groups are $\{1\}$, C_2, C_3, C_4, V_4, D_4, and A_4, as evidenced by the families below. We exclude S_4 as shown in the rest of this section. □

Assume $\phi \in \mathrm{Rat}_3^1$ as in Eq. (20). Let $W := \mathbf{Fix}(\phi)$ be the set of fixed points, and $\sigma \in \mathrm{Aut}(\phi)$ is a non-trivial automorphism of order n. By Lem. 5 we have $n = 2, 3$, or 4. The case of $n = 4$ is a subcase of $n = 2$. Hence, there are two main cases $n = 2$ and $n = 3$.

4.2. **Involutions.** Let $\sigma \in \mathrm{Aut}(\phi)$ such that $|\sigma| = 2$. We can pick a coordinate in $\mathbb{P}^1$ as in Eq. (31) such that $\sigma(z) = -z$. Then $\mathbf{Fix}(\sigma) = \{0, \infty\}$. From Lem. 4, we have that $\phi(0), \phi(\infty) \in \mathbf{Fix}(\sigma)$. Hence, there are two cases: ϕ fixes points of $\mathbf{Fix}(\sigma)$ or ϕ permutes points of $\mathbf{Fix}(\sigma)$ and each of them corresponds to an irreducible surface as we will see next.

4.2.1. *ϕ fixes points of $\mathbf{Fix}(\sigma) = \{0, \infty\}$.* In this case, $\phi(z)$ fixes both points in $\mathbf{Fix}(\sigma)$, so $\phi(0) = 0$ and $\phi(\infty) = \infty$. By Prop. 2, $\phi(z) = z\psi(z^2)$, where $\psi(z^2)$ is a rational function ensuring ϕ is degree 3 with $\sigma(z) = -z$ as an automorphism. Hence,

$$\phi(z) = z\psi(z^2) = \frac{z(z^2 + a)}{z^2 + b},$$

for $a, b \in k$. Assuming $b \neq 0$ (to avoid a pole at $z = 0$), rewrite $\phi(z)$ as

$$\phi(z) = \frac{1}{b} \cdot \frac{z(z^2 + a)}{\frac{1}{b}z^2 + 1} = \frac{1}{b} \cdot \frac{z^3 + az}{\frac{1}{b}z^2 + 1}.$$

Define $t = a$ and $s = \frac{1}{b}$ and we have

$$\phi(z) = \frac{1}{b} \cdot \frac{z^3 + tz}{sz^2 + 1}.$$

Since $\phi = \frac{F(z)}{G(z)}$ is defined up to a scalar multiple, we can scale by b, yielding:

$$\phi(z) = z\frac{z^2 + t}{sz^2 + 1},$$

for $t, s \in k$, with $s \neq 0$. This matches the family in [7]. Then

$$I_6(\phi) = t^2s^2 - 2ts + 1 = (ts - 1)^2 \neq 0.$$

Hence, in this case ϕ is conjugate to a rational function written in the form

$$\boxed{\phi(z) = \frac{z^3 + tz}{sz^2 + 1}, \qquad \mathfrak{p} = [1 : 0 : t : 0 : 0 : s : 0 : 1]} \tag{32}$$

for some $t, s \in k$ such that $ts \neq 1$.

4.2.2. *ϕ permutes points of* $\mathbf{Fix}(\sigma)$. We are still under assumption that there is an involution $\sigma \in \mathrm{Aut}(\phi)$ such that $\sigma(z) = -z$. Assume that ϕ permutes 0 and ∞. From Prop. 2, we have that $\phi(z) = \frac{1}{z}\psi(z^2)$, for some degree two rational function $\psi(z)$. Lem. 4 implies that σ permutes points in $\phi^{-1}(0)$ and $\phi^{-1}(\infty)$ which are the numerator and denominator of ψ. Thus

$$\phi(z) = \frac{1}{z}\frac{z^2 + b}{z^2 + a}.$$

Similarly as the previous case, we can normalize $b = 1$ and get

$$\boxed{\phi(z) = \frac{sz^2 + 1}{z^3 + tz}, \qquad \mathfrak{p} = [0 : s : 0 : 1 : 1 : 0 : t : 0]} \tag{33}$$

for some $t, s \in k$. The resultant in this case is

$$I_6(\phi) = -(ts - 1)^2 \neq 0$$

which implies that $ts \neq 1$.

Remark 5. *Notice that $\phi(z)$ in Eq. (33) is the reciprocal of the function ϕ given in Eq. (32).*

4.3. **Extra involutions.** Assume that there is another involution $\tau \in \mathrm{Aut}(\phi)$. Since σ fixes 0 and ∞, then τ must permute. Hence $\tau(z) = \frac{1}{z}$. Thus we have the Klein 4-group V_4 embedded in the automorphism group $\mathrm{Aut}(\phi)$.

Proposition 3. *Let $\phi(z)$ be a cubic rational function with an involution, written as*

$$\phi(z) = \frac{az^2 + b}{z(cz^2 + d)},$$

where $a, b, c, d \in k$. If $\phi(z)$ has another involution, then $\phi(z)$ is conjugate under $\mathrm{PGL}(2, k)$ to one of:

$$\phi(z) = \frac{tz^2 + 1}{z^3 + tz} \quad \text{or} \quad \phi(z) = \frac{sz^2 - 1}{z^3 - sz},$$

for some $t, s \in k$.

Proof. Let us see what condition on coefficients a, b, c, d are enforced by $\tau(z) = \frac{1}{z}$.

$$\phi(\tau(z)) = \phi\left(\frac{1}{z}\right) = \frac{a\left(\frac{1}{z}\right)^2 + b}{\frac{1}{z}\left(c\left(\frac{1}{z}\right)^2 + d\right)} = \frac{\frac{a}{z^2} + b}{\frac{c}{z^2} + \frac{d}{z}} = \frac{\frac{a+bz^2}{z^2}}{\frac{c+dz^2}{z^3}} = \frac{z(a+bz^2)}{c+dz^2},$$

$$\tau(\phi(z)) = \frac{1}{\frac{az^2+b}{z(cz^2+d)}} = \frac{z(cz^2+d)}{az^2+b}.$$

Then $\phi(\tau(z)) = \tau(\phi(z))$ implies $\frac{a+bz^2}{c+dz^2} = \frac{cz^2+d}{az^2+b}$, which implies

$$abz^4 + (a^2+b^2)z^2 + ab = cdz^4 + (c^2+d^2)z^2 + cd.$$

Hence we must have $ab = cd$ and $a^2+b^2 = c^2+d^2$. We solve this system of equations for b, d in terms of a, c. Recall that $c \neq 0$, otherwise $\deg \phi < 3$. Then we have $(b, d) = (c, a)$ or $(b, d) = (-c, -a)$. If $(b, d) = (c, a)$ we have $\phi(z) = \frac{az^2+c}{cz^3+az} = \frac{tz^2+1}{z^3+tz}$, for $t = a/c$. If $(b, d) = (-c, -a)$ we have $\phi(z) = \frac{az^2-c}{cz^3-az} = \frac{tz^2-1}{z^3-tz}$, for $t = a/c$. Thus, $\phi(z)$ is conjugate to $\frac{tz^2+1}{z^3+tz}$ or $\frac{sz^2-1}{z^3-sz}$. □

4.3.1. *First case.* Let us denote the locus of all $\phi(z)$ is in the first case of the Prop. 3 by $\mathcal{L}_4$. Notice that if $\phi \in \mathcal{L}_2$ this case implies that $t = s$. Thus,

$$\boxed{\phi(z) = \frac{tz^2+1}{z^3+tz}, \quad \mathfrak{p} = [0, t, 0, 1, 1, 0, t, 0]} \tag{34}$$

Similarly one can show that if $\phi \in \mathcal{L}_1$ then we still get $t = s$, which is a confirmation that this locus will be the intersection $\mathcal{L}_1 \cap \mathcal{L}_2$. We will denote it by $\mathcal{L}_4$.

We denote by $u := t^2$. Then I_6 invariant is

$$I_6(\phi) = -(t^2-1) = -(u-1)^2 \neq 0.$$

4.3.2. *Second case.* Let us denote the locus of all $\phi(z)$ is in the first case of the Proposition by $\mathcal{L}_4$. Notice that if $\phi \in \mathcal{L}_2$ this case implies that $t = -s$.

$$\boxed{\phi(z) = \frac{sz^2-1}{z^3-sz}, \quad \mathfrak{p} = [0, s, 0, -1, 1, 0, -s, 0]} \tag{35}$$

The resultant here is $I_6(\phi) = -s(s^2-1)^2 \neq 0$ for $s \neq 0, \pm 1$.

4.3.3. *Alternating group:* $\mathrm{Aut}(\phi) \cong A_4$. Suppose a Klein four subgroup $V_4 \subset \mathrm{Aut}(\phi)$ extends to an A_4–subgroup. From Eq. (31), we may take $\sigma(z) = -z$ and $\tau(z) = \frac{z+i}{z-i}$, which generate a copy of A_4 inside $\mathrm{PGL}_2(k)$. In this case the corresponding rational map is

$$\boxed{\phi(z) = \frac{z^3-3}{-3z^2}}, \qquad \mathbf{p} = [1:0:0:-3:0:-3:0:0]. \tag{36}$$

Since the A_4–action on $\mathbb{P}^1$ is unique up to conjugacy, and the space of degree–3 relative invariants for this action is one–dimensional, the resulting cubic map is unique up to conjugacy. Hence the locus of maps with automorphism group A_4 consists of a single point in $\mathcal{M}_3^1$.

4.3.4. *Dihedral group D_4.* Here, we can assume that σ is an automorphism of the form $\sigma(z) = \zeta_4 z$. where ζ_4 is a fixed primitive 4th root of unity in k, is an automorphism.

Since σ^2 in $\mathrm{Aut}(\phi)$ acts as $\sigma(z) = -z$, a situation we discussed earlier, we can begin with a rational function of the form in $\mathcal{L}_1$ or $\mathcal{L}_2$, which corresponds to

$$\mathfrak{p} = [1:0:t:0:0:s:0:1] \quad \text{or } \mathfrak{p} = [0:t:0:1:1:0:s:0]$$

Verifying that $\sigma \in \mathrm{Aut}(\phi)$, we can confirm that the first case does not yield any possible rational cubic functions, while in the second case, it must be the case that $t = s = 0$. Hence

$$\boxed{\phi = \frac{1}{z^3}, \qquad \mathfrak{p} = [0,0,0,1,1,0,0,0]} \tag{37}$$

4.4. **An automorphism of order 3.** Let σ be of order 3 taken as in Eq. (31), so $\sigma(z) = \zeta_3 z$. Then, by Prop. 2, we have $\langle\sigma\rangle$ acts on the fiber $\phi^{-1}(0)$, which implies that the numerator of $\phi(z)$ is a polynomial $p(z) = z^3 - t$, for some $t \in k^\times$. We can pick 0 and ∞ in $\phi^{-1}(\infty)$. Since $\phi^{-1}(\infty)$ is an orbit of $\langle\sigma\rangle$ and σ fixes them, then one of them must have multiplicity 2. Thus, we can take

$$\boxed{\phi(z) = \frac{z^3 - t}{z}, \quad \mathfrak{p} = [1:0:0:-t:0:0:1:0]} \tag{38}$$

for some $t \in k$. The resultant is $I_6(\phi) = -t^3$, which gives the condition that $t \neq 0$.

	G	$\phi(z)$	$\mathfrak{p} \in \mathbb{P}^7$	dim	Eq. $\mathcal{L}_i$	$\xi(\phi)$
$\mathcal{L}_1$	C_2	$\frac{z^3+tz}{sz^2+1}$	$[1,0,t,0,0,s,0,1]$	2	(40)	$\xi_2 = \xi_3 = 0$
$\mathcal{L}_2$	C_2	$\frac{sz^2+1}{z^3+tz}$	$[0,s,0,1,1,0,t,0]$	2	(41)	$\xi_5 = 0$
$\mathcal{L}_3$	C_3	$\frac{z^3-t}{z}$	$[1,0,0,-t,0,0,1,0]$	1	(48)	
$\mathcal{L}_4$	V_4	$\frac{tz^2+1}{z^3+tz}$	$[0,t,0,1,1,0,t,0]$	1	(43)	$\xi_2 = \xi_3 = \xi_5 = 0$
$\mathcal{L}_5$	V_4	$\frac{tz^2-1}{z^3-tz}$	$[0,t,0,-1,1,0,-t,0]$	1	(44)	$\xi_0 = \xi_3 = \xi_4 = \xi_5 = 0$
$\mathcal{L}_6$	A_4	$\frac{z^3-3}{-3z^2}$	$[1,0,0,-3,0,-3,0,0]$	0	(46)	$[0:0:18:0:0:0]$
$\mathcal{L}_7$	D_4	$\frac{1}{z^3}$	$[0,0,0,1,1,0,0,0]$	1	(45)	$[0:-2:0:0:0:0]$

TABLE 1. Automorphism loci of degree 3 rational functions

This completes all the cases. We summarize all cases in Table 1. These results agree with results in [7]. Justifying the last two columns of the table will be the focus of the next section.

Inclusions among the loci are described in diagram Fig. 1. Notice that in each node of the diagram Fig. 1 we also put the invariants which vanish in that locus. Justifying these inclusions , computing the invariants, and describing each locus in terms of these invariants will be done in the next section.

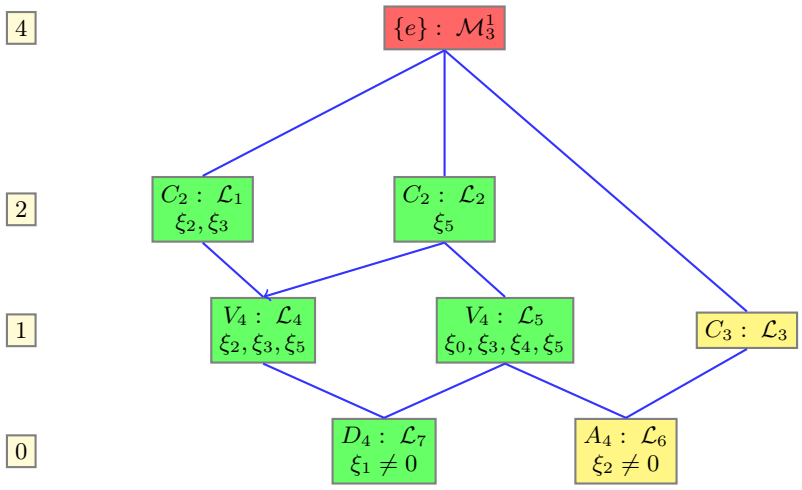


FIGURE 1. The inclusions among the loci for degree 3 rational functions

Remark 6. *Those familiar with automorphisms groups of hyperelliptic curves notice the similarity between the two problems. The zeroes and poles of $\phi(z)$ now play the role of the Weierstrass points of the hyperelliptic curve. While the reduced automorphism group of a hyperelliptic curve acts on the set of Weierstrass points, in our case here* $\mathrm{Aut}(\phi)$ *acts on the set of zeroes of $\phi(z)$ and on the set of poles. A general treatment of automorphism group of $\phi(z)$ can be done similarly to that of hyperelliptic curves; see* [13].

5. Computation of the Loci with Fixed Automorphism Group

In this section, we compute the loci of rational cubics with a fixed automorphism group in terms of invariants of rational cubics. We follow an analogy with algebraic curves and build on previous work by the third author in computing similar loci in the moduli space of genus 2 curves; see [8, 13–15]. The first question is where this locus lies and what type of invariants we should use to compute it. Let $\mathcal{L}$ denote the image in $\mathbb{P}_{(2,2,3,3,4,6)}$ of the map

$$\Phi : (c_0, \ldots, c_8) \to (\xi_0(c_0, \ldots, c_8), \ldots, \xi_5(c_0, \ldots, c_8)) .$$

This image is a weighted projective variety in $\mathbb{P}_{(2,2,3,3,4,6)}$. Alternatively, one can use the Veronese morphism in Eq. (24) and compute the locus of such rational functions as a projective variety, or even as an affine variety in terms of absolute invariants $i_1, \ldots, i_5$.

Computing these loci in terms of $i_1, \ldots, i_5$ is straightforward but highly inefficient. We could express $i_1, \ldots, i_5$ in terms of the parameters described for each family in the previous section and then use a Gröbner basis approach to eliminate those parameters. However, the degrees of the resulting equations are very large, making this approach computationally challenging even for the relatively simple families considered here. A more efficient method is to compute the loci as subvarieties of weighted projective spaces. This requires a modification of Buchberger's algorithm to handle computations in a weighted projective space.

5.1. **Gröbner Bases for Weighted Homogeneous Ideals.** Consider a subvariety in $\mathbb{P}(w_0, \ldots, w_n)$ defined parametrically by a set of polynomials over m parameters. The coordinates are given by:

$$x_i = \xi_i, \quad i = 0, \ldots, n,$$

where each ξ_i is a polynomial in $k[t_1, \ldots, t_m]$, and $t_1, \ldots, t_m$ are parameters with assigned degrees (typically positive integers). The image of this parameterization is a variety whose dimension depends on m; for m independent parameters, the dimension is typically $m - 1$. To define this variety explicitly, we seek polynomial relations among $x_0, \ldots, x_n$ that hold for all parameter values, and these relations must be weighted homogeneous with respect to the weights $(w_0, \ldots, w_n)$. In other words, the relations must be satisfied not only by tuples $(\xi_0, \ldots, \xi_n)$ but also by all $(\lambda^{w_0}\xi_0, \ldots, \lambda^{w_n}\xi_n)$.

To compute these relations, we construct the ideal $I = \langle x_0 - \xi_0, x_1 - \xi_1, \ldots, x_n - \xi_n \rangle$ in the polynomial ring $k[t_1, \ldots, t_m, x_0, \ldots, x_n]$. The defining equations of the parameterized variety's image in $\mathbb{P}(w_0, \ldots, w_n)$ are the elements of the elimination ideal $I \cap k[x_0, \ldots, x_n]$, which consists of all polynomials in $x_0, \ldots, x_n$ satisfied by the parameterization. By choosing a monomial order that eliminates the parameters $t_1, \ldots, t_m$, the Gröbner basis of I includes polynomials free of t_j, representing the desired relations.

To facilitate elimination while respecting the weighted structure, we assign degrees to both parameters and coordinates, say $\deg(t_j) = d_j$ and $\deg(x_i) = w_i$. Typically, we set $d_j = 1$.

The weighted degree lexicographic order (`wdeglex`) compares monomials first by their total weighted degree, then lexicographically, with parameters ordered above coordinates (e.g., $t_1 > \cdots > t_m > x_0 > \cdots > x_n$). This ensures that the Gröbner basis computation prioritizes elimination of the parameters, yielding a set of generators for $I \cap k[x_0, \ldots, x_n]$ that define the variety's image in the coordinate ring of $\mathbb{P}(w_0, \ldots, w_n)$.

The relations obtained from the elimination ideal are algebraically correct but may not be weighted homogeneous, as their terms can have differing weighted degrees. To make them valid defining equations in $\mathbb{P}(w_0, \ldots, w_n)$, a homogenization step is required:

(1) **Degree Computation**: For a relation $F = \sum c_\alpha x^\alpha$, compute the weighted degree of each term as $\deg(x^\alpha) = \sum_{i=0}^{n} w_i \alpha_i$.
(2) **Target Degree**: Determine a common degree d, typically the least common multiple (LCM) of the degrees of all terms in F, i.e., $d = \mathrm{LCM}(\deg(x^\alpha))$.
(3) **Exponent Adjustment**: Transform each term $c_\alpha x^\alpha$ into $c_\alpha x^{\alpha k_\alpha}$, where $k_\alpha = d/\deg(x^\alpha)$, ensuring $\deg(x^{\alpha k_\alpha}) = k_\alpha \cdot \deg(x^\alpha) = d$. Adjust coefficients as needed to preserve the relation's zero set.

This process yields a weighted homogeneous polynomial that defines the same variety but adheres to the grading of $\mathbb{P}(w_0, \ldots, w_n)$.

5.1.1. *General Algorithm.* For a weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$ with weights $w = (w_0, \ldots, w_n)$ and a parameterization $x_i = \xi_i(t_1, \ldots, t_m)$:

(1) Construct the ideal $I = \langle x_0 - \xi_0, x_1 - \xi_1, \ldots, x_n - \xi_n \rangle$ in $k[t_1, \ldots, t_m, x_0, \ldots, x_n]$.
(2) Compute the Gröbner basis B of I using a weighted degree lexicographic order with weights $(d_1, \ldots, d_m, w_0, \ldots, w_n)$ and order $t_1 > \cdots > t_m > x_0 > \cdots > x_n$.

(3) Extract the set $R = \{f \in B \mid f \in k[x_0, \ldots, x_n]\}$, the generators of the elimination ideal.
(4) For each polynomial $f \in R$:
    i) If f is single-term or all terms have the same weighted degree, retain f as-is.
    ii) Otherwise, compute $d = \mathrm{LCM}(\deg(x^\alpha))$ over all terms in f, and adjust f to $F = \sum c_\alpha x^{\alpha k_\alpha}$, where $k_\alpha = d/\deg(x^\alpha)$, ensuring F is weighted homogeneous of degree d.
(5) Define the ideal $J = \langle R_{\mathrm{homog}} \rangle$ in $k[x_0, \ldots, x_n]$, which specifies the subvariety in $\mathbb{P}(w_0, \ldots, w_n)$.

This method uses Gröbner bases to eliminate parameters and enforces weighted homogeneity, providing a computational framework for studying subvarieties in weighted projective spaces with arbitrary weights $(w_0, \ldots, w_n)$. The method is described in [6] and related works by its authors.

In our situation, however, there is an additional consideration. Our invariants $\xi_0, \ldots, \xi_5$ are homogeneous polynomials when applied to the definition of ϕ in Eq. (5), meaning they are homogeneous in $c_0, \ldots, c_7$. However, they are not necessarily homogeneous with respect to our specializations for t and s from the previous section. Thus, we first homogenize these defining equations by introducing another variable, say t_0, and then apply the above procedure.

Next, we explicitly compute the loci as weighted projective varieties for each case, illustrating the approach described above.

5.2. **Locus $\mathcal{L}_1$.** We assume that $\phi(z)$ is given as in Eq. (32). Computing the invariants $\xi(\phi)$, we obtain:

$$\begin{aligned}
&\xi_0 = 2(t+3)(s+3), \quad \xi_1 = \frac{1}{2}(t-1)(s-1), \quad \xi_2 = \xi_3 = 0,\\
&\xi_4 = -\frac{1}{3}\left(t^2s^2 + 2t^2s + 3t^2 + 2ts^2 - 8ts - 6t + 3s^2 - 6s + 9\right),\\
&\xi_5 = -(t-s)^2(ts + t + s - 3).
\end{aligned}$$

There is an involution permuting t and s. Let u and v denote invariants of that involution, defined as:

$$\boxed{u = t + s \quad \text{and} \quad v = ts.}$$

We can express all invariants in terms of u and v as follows:

$$\begin{aligned}
&\xi_0 = 2(v + 3u + 9), \quad \xi_1 = \frac{1}{2}(v - u + 1), \quad \xi_2 = \xi_3 = 0,\\
&\xi_4 = -\frac{1}{3}\left(v^2 + 3u^2 + 2uv - 14v - 6u + 9\right),\\
&\xi_5 = (u^2 - 4v)(3 - u - v).
\end{aligned}$$

Also, $I_6 = (ts-1)^2 = (v-1)^2 \neq 0$, so $v \neq 1$. Moreover, J_6 is given by:

$$J_6 = 4(s+3)(t+3)(st + s + t - 3)^2 = 4(3u + v + 9)(u + v - 3)^2. \tag{39}$$

By computing a Gröbner basis for the weighted homogeneous system as described above, we obtain the following degree 24 weighted hypersurface:

(40)
$$\begin{aligned}\mathcal{L}_1: \quad & \xi_0^6\xi_1^6 - 54\xi_0^4\xi_1^4\xi_4^2 - \frac{27}{4}\xi_0^4\xi_1^3\xi_5^2 - 27\xi_0^3\xi_1^4\xi_5^2 - 108\xi_0^3\xi_1^3\xi_4^3 + 729\xi_0^2\xi_1^2\xi_4^4 + \frac{729}{4}\xi_0^2\xi_1\xi_4^2\xi_5^2 \\ & + \frac{729}{64}\xi_0^2\xi_5^4 + 729\xi_0\xi_1^2\xi_4^2\xi_5^2 + 2916\xi_0\xi_1\xi_4^5 + \frac{243}{8}\xi_0\xi_1\xi_5^4 + \frac{729}{2}\xi_0\xi_4^3\xi_5^2 + \frac{729}{4}\xi_1^2\xi_5^4 \\ & + 1458\xi_1\xi_4^3\xi_5^2 + 2916\xi_4^6 + \frac{729}{2}\xi_4\xi_5^4 = 0\end{aligned}$$

We can also compute $\mathcal{L}_1$ in terms of the absolute invariants by eliminating u and v. These computations are lengthy but show that, on the open subset where $J_6 \neq 0$, both u and v can be expressed as rational functions of (i_1, i_2, i_5). Thus the parametrization induces a rational map

$$\Phi_1 : k^2 \setminus \{v = 1\} \longrightarrow \mathcal{L}_1 \subset k^3, \qquad (u, v) \longmapsto (i_1, i_2, i_5),$$

whose image is exactly the hypersurface $\mathcal{L}_1$, and which is birational with rational inverse given by the functions $u = u(i_1, i_2, i_5)$ and $v = v(i_1, i_2, i_5)$ on $L_1 \cap \{J_6 \neq 0\}$.

Remark 7. *The invariants $u = t+s$ and $v = ts$ mirror dihedral invariants in genus 2 curves [8,15], symmetric under swapping t and s. Here, they are invariant under parameter rescaling and conjugation adjusting t and s, simplifying computations akin to symmetric polynomials in root permutations—a pattern likely extending to higher-degree rational functions.*

If $\xi_0 = 0$, then $t = -3$ or $s = -3$, and the moduli point is: $\xi(\phi) = [0 : 8 : 0 : 0 : 0 : 0]$, a singular point in $\mathcal{L}_1$ with automorphism group isomorphic to D_4 (cf. Eq. (45)).

Remark 8. *Note that the factor $u^2 = 4v$ (or equivalently $t = s$) makes $\xi_5 = 0$. This locus corresponds to $\mathcal{L}_4$ (cf. Eq. (43)), where $\phi(z)$ has an extra involution. Since $\mathcal{L}_4$ lies in the intersection of $\mathcal{L}_1$ and the next locus, we will discuss it in detail later.*

5.3. **Locus $\mathcal{L}_2$.** Assume that ϕ is given as in Eq. (33). We follow the same approach as above. Computing $\xi(\phi)$, we get:

$$\begin{aligned}&\xi_0 = -2(t+s)^2, \quad \xi_1 = \frac{1}{6}\left((t-s)^2 - 12\right), \quad \xi_2 = -\frac{1}{36}(s-t)\left((t-s)^2 + 36\right), \\ &\xi_3 = \frac{2}{3}(s-t)(t+s)^2, \quad \xi_4 = -\frac{1}{9}(t+s)^2\left((s-t)^2 + 12\right), \quad \xi_5 = 0.\end{aligned}$$

We define invariants:

$$u := (t+s)^2 \quad \text{and} \quad v := s - t,$$

and express the invariants in terms of u and v as follows:

$$\begin{aligned}&\xi_0 = -2u, \quad \xi_1 = \frac{1}{6}(v^2 - 12), \quad \xi_2 = -\frac{1}{36}v(v^2 + 36), \\ &\xi_3 = \frac{2}{3}uv, \quad \xi_4 = -\frac{1}{9}u(v^2 + 12), \quad \xi_5 = 0.\end{aligned}$$

Here, we have:

$$J_6(\phi) = -16(s+t)^4 = -16u^2 \quad \text{and} \quad I_6 = \frac{v^2 - u + 4}{4}.$$

Using a Gröbner basis to eliminate u and v, we determine the equations for $\mathcal{L}_2$ in $\mathbb{P}(2,2,3,3,4,5)$ as follows:

$$\mathcal{L}_2 : \quad \begin{cases} \xi_5 = 0, \\ \xi_0^2\xi_1 + 3\xi_0\xi_4 - 3\xi_3^2 = 0, \\ \xi_0^2\xi_2 + \frac{1}{2}\xi_0\xi_1\xi_3 - 3\xi_3\xi_4 = 0, \\ \xi_0\xi_1\xi_4 - \xi_0\xi_2\xi_3 - \frac{1}{2}\xi_1\xi_3^2 + 3\xi_4^2 = 0 \end{cases} \tag{41}$$

The absolute invariants are:

$$i_1 = \frac{1024u^6}{(-v^2+u-4)^2}, \quad i_2 = \frac{(v^2-12)^6}{2916(-v^2+u-4)^2}, \quad i_3 = \frac{v^4(v^2+36)^4}{104976(-v^2+u-4)^2},$$
$$i_4 = \frac{256v^4u^4}{81(-v^2+u-4)^2}, \quad i_5 = -\frac{16u^3(v^2+12)^3}{729(-v^2+u-4)^2}.$$

Assuming $J_6 \neq 0$ (i.e., $u \neq 0$), we eliminate u and v, finding:

$$u = -\frac{1}{2}\xi_0, \quad v = 3\frac{\xi_3}{\xi_0}. \tag{42}$$

Similarly to the previous case, the parameters (u, v) give a rational parametrization of the locus $\mathcal{L}_2$ of maps with automorphism group C_3. Eliminating u and v from the expressions for the absolute invariants $(i_1, \ldots, i_5)$ yields the defining equation of $\mathcal{L}_2$ as a hypersurface in k^3 (or in k^5 when all absolute invariants are retained). On the open subset where $J_6 \neq 0$, both u and v can be recovered as rational functions of the absolute invariants, so the parametrization induces a birational map

$$\Phi_2 : k^2 \dashrightarrow \mathcal{L}_2, \qquad (u, v) \longmapsto (i_1, i_2, i_5).$$

Thus $\mathcal{L}_2$ is an irreducible surface (2-dimensional variety) birationally parametrized by (u, v), and the field of moduli of $\phi(z)$ is $k(u, v)$.

Consider now the case when:

$$J_6 = -16(s+t)^4 = -16u^2 = 0,$$

implying $t = -s$. The function becomes:

$$\phi(z) = \frac{sz^2+1}{z^3 - sz},$$

and then $\xi_0 = \xi_3 = \xi_4 = \xi_5 = 0$, $\xi_1 = \frac{v^2}{6} - 1$, and $\xi_2 = -v$. This corresponds to $\mathcal{L}_5$, as we will see later.

This concludes the loci $\mathcal{L}_1$ and $\mathcal{L}_2$ for rational functions with involutions as described in Section 4.2. The remaining loci are of dimension one or zero, making their computations simpler. We will address each in detail below.

5.4. **Locus $\mathcal{L}_4$.** Assume ϕ is as in Eq. (34), with homogeneous parameterization in $\mathbb{P}_{(2,2,3,3,4,6)}$:

$$\xi(\phi) = \left[-8t^2,\ -2u^2,\ 0,\ 0,\ -\frac{16t^2u^2}{3},\ 0\right].$$

In the affine patch $(u = 1)$:

$$\xi(\phi) = \left[-8t^2,\ -2,\ 0,\ 0,\ -\frac{16t^2}{3},\ 0\right].$$

The Gröbner basis with weighted degree lexicographic order is:

$$\mathcal{L}_4: \quad \begin{cases} \xi_0\xi_1 + 3\xi_4 = 0, \\ \xi_2 = \xi_3 = \xi_5 = 0 \end{cases} \tag{43}$$

This defines a 1-dimensional variety in $\mathbb{P}_{(2,2,3,3,4,6)}$.

5.5. **Locus $\mathcal{L}_5$.** Assume ϕ as in Eq. (35). Its invariants are

$$\xi(\phi) = \left[0,\ \frac{2(s^2+3)}{3},\ -\frac{2s(s-3)(s+3)}{9},\ 0,\ 0,\ 0\right]$$

and $I_6 = (s-1)^2(s+1)^2$. Here we use the absolute invariants i_2 and i_3 and have the following system

$$\begin{cases} i_2\, s^4 - 2i_2\, s^2 + i_2 - \dfrac{4}{9}s^4 - \dfrac{8}{3}s^2 - 4 = 0 \\ i_3\, s^4 - 2i_3\, s^2 + i_3 - \dfrac{4}{81}s^6 + \dfrac{8}{9}s^4 - 4s^2 = 0 \end{cases}$$

By taking the resultant of these two polynomials with respect to t we get the affine version of this 1-dimensional variety

$$16i_2^3 - 72i_2^2 i_3 + 81 i_2\, i_3^2 - 96 i_2^2 + 216 i_2 i_3 - 36 i_3^2 + 144 i_2 - 96 i_3 - 64 = 0$$

By replacing for i_2 and i_3 their definitions we get a degree 18 weighted hypersurface

$$\begin{aligned} \mathcal{L}_5: \quad & 72\xi_1^6\xi_2^2 - 16\xi_1^9 + 96\xi_1^6 I_6 - 81\xi_1^3\xi_2^4 - 216\xi_1^3\xi_2^2 I_6 - 144\xi_1^3 I_6^2 + 36\xi_2^4 I_6 \\ & + 96\xi_2^2 I_6^2 + 64 I_6^3 = 0 \end{aligned} \tag{44}$$

5.6. **Locus $\mathcal{L}_7$.** Assume ϕ as in Eq. (37). Its invariants are

$$\xi(\phi) = [0, -2, 0, 0, 0, 0] \equiv [0, 1, 0, 0, 0, 0] \tag{45}$$

Here we write $\mathcal{L}(G)$ for the locus of degree-3 rational maps whose full automorphism group contains a subgroup conjugate to $G \subset \mathrm{PGL}_2(k)$. In particular, $\mathcal{L}(C_4)$ and $\mathcal{L}(D_4)$ denote the loci corresponding to the cyclic and dihedral subgroups of order 4. We also let $\sigma_0(z) = -1/z$, the involution generating the order-2 subgroup of S_4 that is not contained in any copy of $D_4 \subset S_4$.

For the map $\varphi(z) = y^3/x^3 = z^3$ in normalized coordinates, the involution $\sigma'(z) = 1/z$ acts as an additional automorphism, so the loci $\mathcal{L}(C_4)$ and $\mathcal{L}(D_4)$ coincide. On the other hand, the involution $\sigma_0(z) = -1/z$, which would be required to extend D_4 to S_4, does not preserve $\varphi(z)$, and therefore no cubic rational map admits S_4 as a subgroup of its automorphism group. Hence $\mathcal{L}(S_4)$ is empty.

5.7. **Locus $\mathcal{L}_6$.** Assume ϕ as in Eq. (34). The moduli point is

$$\mathcal{L}_6: \quad \xi(\phi) = [0:0:18:0:0:0] \equiv [0:0:1:0:0:0] \tag{46}$$

Furthermore, it should be noticed that in this case $J_6(\phi) = 0$.

5.8. **Locus $\mathcal{L}_3$.** Assume ϕ as in Eq. (38). Next, we compute its invariants

$$\xi(\phi) = \left[-2,\ \frac{1}{6},\ \frac{27t+2}{72},\ -\frac{27t+2}{3},\ \frac{54t-1}{9},\ -\frac{t(27t-16)}{4}\right]$$

Notice that $I_6 = t \neq 0$. The absolute invariants are

$$i_1 = \frac{64}{t^6},\ i_2 = \frac{1}{46656t^6},\ i_3 = \frac{(27t+2)^4}{26873856t^2},\ i_4 = \frac{(27t+2)^4}{81t^2},\ i_5 = \frac{(54t-1)^3}{729t^2}$$

By eliminating t from these equations, we can express t as a rational function in terms of i_1, i_5 and get the following affine curve

(47)
$$\begin{aligned}
&61478762617650839961 6i_1i_5^6 + 44264709084708604772352i_1i_5^5 + 49589822592i_1^2i_5^3\\
&+ 11508824362024237240811 52i_1i_5^4 - 6248317646592i_1^2i_5^2 - 4760622968832i_1^2\\
&+ i_1^3 + 35704672266240i_1^2i_5 + 59491769009848364814041088i_1i_5^2\\
&+ 79322358679797819752054784i_1i_5 + 75545103504569352144814 08i_1\\
&- 39960194991849297431698185813586083 84 + 12984314664847857399889920i_1i_5^3 = 0
\end{aligned}$$

We can express this as a weighted projective curve by substituting for i_1 and i_5:

(48)
$$\begin{aligned}
&I_6^4\xi_0^9 - 2834352I_6^3\xi_0^6\xi_4^3 + 24794911296\xi_0^3\xi_4^9 + 3779136I_6^5\xi_0^6 + 892616806656I_6^2\xi_0^3\xi_4^6\\
&+ 7140934453248I_6^4\xi_0^3\xi_4^3 + 47606229688 32I_6^6\xi_0^3 + 19990046271044321 28I_6^7 = 0
\end{aligned}$$

This completes all the cases.

6. A Database of Cubic Rational Functions

In light of the recent success of machine-learning techniques in algebraic geometry — particularly the neuro-symbolic classification of genus-two curves and their isogeny graphs [12], and the earlier systematic enumeration of rational points on moduli spaces [3] — we apply the same methodology to the moduli space $\mathcal{M}_3^1$ of degree-three rational functions. Determining automorphism groups (and more generally the stratification of $\mathcal{M}_d^1$) becomes increasingly subtle as the degree d grows, and the raw coefficient space $\mathrm{Rat}_3^1 \subset \mathbb{P}^7$ suffers from massive redundancy under PGL_2-action. Following the approach that proved highly effective for hyperelliptic curves, we therefore construct a large, clean database of rational cubics over $\mathbb{Q}$ of naive height at most 4, represented directly by their weighted projective invariants $\xi_0, \ldots, \xi_5$ in $\mathbb{P}^5_{(2,2,3,3,4,6)}(\mathbb{Q})$. This database eliminates all conjugacy redundancies, provides a natural low-dimensional feature set, and serves as a testbed for automated classification of automorphism groups (and later for minimal fields of definition and inclusion relations among strata).

In this section, we construct a comprehensive database of cubic rational functions over the rational numbers $\mathbb{Q}$, denoted Rat_3^1, leveraging the weighted projective space $\mathbb{P}^5_\omega(\mathbb{Q})$ with weights $\omega = (2,2,3,3,4,6)$ as a parametrization framework. This database, denoted $\mathcal{P}_3^h$, catalogs rational functions $\phi(x) = \frac{f_0(x)}{f_1(x)} \in \mathbb{Q}(x)$ of degree 3, where $f_0(x)$ and $f_1(x)$ are polynomials of degrees 3 and 2, respectively, ensuring the degree of the rational function is $\deg(\phi) = \deg(f_0) - \deg(f_1) = 3$. Each function is represented as a projective point in $\mathbb{P}^7_{\mathbb{Q}}$, and we impose constraints on height and coprimality to define the dataset systematically.

A cubic rational function $\phi(x)$ is given as in Eq. (20) and corresponds to a point $P_\phi = [c_0 : c_1 : \cdots : c_7] \in \mathbb{P}^7_{\mathbb{Q}}$. We define the naive height of ϕ as

$$H(\phi) = \max\{|c_i| \mid i = 0, \ldots, 7\},$$

and restrict our dataset to functions with $H(\phi) \leq h$, where h is a specified height bound. To ensure well-definedness, we require that the coefficients are coprime, i.e., $\gcd(c_0, c_1, \ldots, c_7) = 1$, and that the resultant $I_6(\phi) = \mathrm{Res}(f_0, f_1) \neq 0$, guaranteeing that ϕ has no common roots between numerator and denominator. Thus, we define:

$$\mathcal{P}_3^h := \{P_\phi \in \mathbb{P}^7_{\mathbb{Q}} \mid H_{\mathbb{Q}}(P_\phi) \leq h,\ \gcd(c_0, \ldots, c_7) = 1,\ I_6(\phi) \neq 0\}.$$

For each $\phi \in \mathcal{P}_3^h$, we compute several invariants and properties to enrich the database. These include the automorphism group $\mathrm{Aut}(\phi)$, determined by checking against the classification of possible groups for cubic rational functions (e.g., $\{e\}$, C_2, D_4, etc.) as outlined in Table 1, the invariants $\mathbf{p} = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ in $\mathbb{P}^5_\omega(\mathbb{Q})$, and the absolute invariants $(i_1, i_2, i_3, i_4, i_5)$. Additionally, we calculate the invariant $J_6(\phi)$ and the weighted moduli height $\hat{h}(\phi)$, which measures the height of the invariants in the weighted projective space; as discussed in detail in [2, 11]. In our case, for a given ϕ we compute $\mathbf{p} = [\xi_0(f), \xi_1(f), \xi_2(f), \xi_3(f), \xi_4(f), \xi_5(f)] \in \mathbb{P}^5_\omega(\mathbb{Q})$ defined over a number field K and its **weighted moduli height** is

$$\hat{h}(\phi) = \prod_v \max_{0 \leq i \leq 5} \left\{ |\xi_i|_v^{1/w_i} \right\}, \qquad (w_0, w_1, w_2, w_3, w_4, w_5) = (2, 2, 3, 3, 4, 6),$$

where the product is over all places v of K. The data is stored in a Python dictionary, with keys given by the coefficient tuples $(c_0, c_1, \ldots, c_7)$ and values as lists containing:

- $H(\phi)$: the naive height,
- $\mathbf{p} = (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$: the projective invariants,
- $\hat{h}(\phi)$: the weighted moduli height,
- $J_6(\phi)$: an additional invariant,
- $\mathrm{Aut}(\phi)$: the automorphism group (e.g., '{e}'),
- $(i_1, i_2, i_3, i_4, i_5)$: the absolute invariants.

An example entry from the database is:

$$(2, 3, -1, -3, 1, 2, -3, 1) \mapsto [3, (32, 12, 13, -164, -424, 2572), 5.66, 89360, '\{e\}',$$
$$\left(\frac{1073741824}{44521}, \frac{2985984}{44521}, \frac{531441}{712336}, \frac{723394816}{44521}, -\frac{76225024}{44521}\right)].$$

Here, the key $(2, 3, -1, -3, 1, 2, -3, 1)$ represents the coefficients of

$$\phi(x) = \frac{2 + 3x - x^2 - 3x^3}{1 + 2x - 3x^2 + x^3},$$

with $H(\phi) = 3$, invariants ξ_i mapping to a point in $\mathbb{P}^5_\omega(\mathbb{Q})$, a weighted height of 5.66, $J_6 = 89360$, the trivial automorphism group $\{e\}$, and the corresponding absolute invariants.

To illustrate the scope of the dataset, Table 2 shows the distribution of automorphism groups for rational cubics of weighted moduli height $H(\phi) \leq 3$. The entries are grouped according to the loci $\mathcal{M}_3^1, \mathcal{L}_1, \ldots, \mathcal{L}_7$, corresponding respectively to the trivial group $\{e\}$ and the nontrivial automorphism groups C_2, C_3, D_4, and so on, as classified earlier in the paper.

TABLE 2. Distribution of groups for cubics of height $H(\phi) \leq 3$

H	$\mathcal{M}_3^1$	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$	$\mathcal{L}_5$	$\mathcal{L}_6$	$\mathcal{L}_7$	Total
1	2223	9	8	6	0	0	0	2	2248
2	84267	34	12	17	0	0	0	2	84332
3	814126	81	66	44	1	22	18	50	814408
Total	900616	124	86	67	1	22	18	54	900988

The distribution is strongly skewed toward the trivial automorphism group $\{e\}$ (the locus $\mathcal{M}_3^1$), which accounts for 900,616 of the 900,988 distinct conjugacy classes of height $H(\phi) \leq 3$. Nontrivial groups such as D_4 ($\mathcal{L}_3$) and A_4 ($\mathcal{L}_7$) occur much less frequently, in accordance with their rarity among cubic rational maps.

The case $H(\phi) = 4$ is substantially more demanding computationally. Our current search has identified at least 350,679 distinct moduli points in $\mathcal{M}_3^1$ of weighted moduli height 4, but the enumeration is not yet certified to be complete. We therefore record 350,679 as a rigorous lower bound for the number of conjugacy classes of height 4. The fully completed data for heights $0 \leq h \leq 3$ appear in Table 2.

This database provides a robust foundation for subsequent analysis, including the machine learning classification of automorphism groups discussed in Section 8. The inclusion of both projective and absolute invariants, alongside height metrics and group labels, enables a detailed exploration of the geometric and arithmetic properties of Rat_3^1.

7. Classification of Groups Using Machine Learning

The machine–learning experiments in this section are intended solely as an illustrative exploration of how the absolute invariants may be used within a computational classification framework. No mathematical result in the paper depends on these experiments, and their purpose is not to establish statistical performance guarantees but to demonstrate that the invariant–theoretic structure developed in the earlier sections lends itself naturally to algorithmic methods. The extreme class imbalance visible in small-height data is intrinsic to the geometry of the moduli space and not a feature of the classifier. Accordingly, the numerical results below should be interpreted as qualitative evidence rather than as a benchmark evaluation of machine–learning models.

Automorphism Group	Proportion of Data	Samples in Test Set
{e} (6)	99.84%	179,910
C2-2 (2)	0.12%	214
C2-1 (1)	0.02%	29
D4 (3)	0.01%	16
V4-1 (4)	0.0075%	17
A4 (0)	0.0067%	12
V4-2 (5)	0.0002%	2

TABLE 3. Class distribution in the dataset.

Our experiments aim to classify the automorphism group of a rational cubic function by means of supervised learning. Two approaches were tested: a baseline

model using the coefficients of the rational function as input features, and a refined model using the weighted invariants.

The dataset used here consists of rational functions of weighted moduli height $H(\phi) \leq 4$, partitioned by automorphism group. Its distribution is extremely imbalanced: the generic case $\{e\}$ accounts for over 99% of all samples, reflecting the fact that rational cubics with nontrivial automorphism group form very sparse loci in $\mathcal{M}_3^1$. Table 3 lists the proportions and the number of samples appearing in the testing set for each group.

7.1. **Initial Model: Using Coefficients as Features.** We formulate the classification of automorphism groups as a supervised learning problem, where the input features are the coefficients of the rational functions and the target variable is the corresponding automorphism group. A Random Forest classifier with 100 estimators was trained on these features.

The reported overall accuracy of 99.97% is not meaningful in this setting, because the dataset is dominated by the generic class $\{e\}$, which accounts for 99.84% of all samples. A classifier that always predicts $\{e\}$ would already achieve almost the same accuracy. More informative performance metrics appear in Table 4. These show that, although the majority class is classified perfectly (precision, recall, and F1-score all equal to 1.00), the minority classes exhibit very low recall—for instance, 0.14 for $C2$–1 and 0.41 for $V4$–1—indicating that the model frequently collapses the rare strata into the dominant class. This behavior reflects the fact that raw coefficients do not encode the moduli-theoretic structure of $\mathcal{M}_3^1$ and are therefore a poor feature set for distinguishing automorphism groups.

Class	Precision	Recall	F1-score
A4 (0)	1.00	0.58	0.74
C2-1 (1)	1.00	0.14	0.24
C2-2 (2)	0.97	0.91	0.94
D4 (3)	1.00	0.94	0.97
V4-1 (4)	0.78	0.41	0.54
{e} (6)	1.00	1.00	1.00

TABLE 4. Performance metrics using coefficients as features.

7.1.1. *Addressing Class Imbalance.* To partially mitigate the class imbalance, we apply class weighting to the Random Forest classifier. For each class i we assign a weight $w_i = \frac{N}{C \times n_i}$, where N is the total number of samples, C the number of classes, and n_i the number of samples in class i. This increases the influence of underrepresented classes during training.

After retraining with these weights, the reported accuracy remains 99.96%, but the performance across classes becomes more informative. As shown in Table 5, recall improves for some minority groups—for example, $A4$ increases from 0.58 to 0.83—while other rare classes remain difficult to identify (e.g. $C2$–1 decreases to 0.10 and $V4$–1 to 0.29). This indicates that class weighting reduces, but does not eliminate, the limitations inherent in using coefficients rather than invariant-based representations.

Class	Precision	Recall	F1-score
A4 (0)	1.00	0.83	0.91
C2-1 (1)	1.00	0.10	0.19
C2-2 (2)	0.98	0.90	0.94
D4 (3)	1.00	0.94	0.97
V4-1 (4)	0.71	0.29	0.42
{e} (6)	1.00	1.00	1.00

TABLE 5. Performance metrics after applying class weighting to the coefficient-based model.

7.2. **Using Invariants as Input Features.** Given the limitations of the coefficient-based approach—particularly its difficulty distinguishing minority strata even after class weighting—we train a second model using the weighted invariants introduced in Sections 4–6 as input features, following the approach of graded neural networks in [16]. These invariants provide a complete moduli-theoretic description: by Lemma 3, they determine the conjugacy class of a rational cubic uniquely. Thus, unlike raw coefficients, they encode exactly the structure relevant for identifying the automorphism group.

A Random Forest classifier with 100 estimators was trained using the same training–testing split and the same class-weighting scheme as in the previous subsection. The results, summarized in Table 6, show a substantial improvement across all minority classes. The reported accuracy

$$\text{Reported Accuracy} = 0.9999223076837701 \approx 99.992\%$$

should again be interpreted in light of the strong class imbalance; however, in contrast to the coefficient-based model, the classwise precision and recall here provide meaningful information because the invariants separate strata in $\mathcal{M}_3^1$.

Class	Precision	Recall	F1-score	Support
A4 (0)	1.00	1.00	1.00	12
C2-1 (1)	1.00	0.93	0.96	29
C2-2 (2)	1.00	0.95	0.98	214
D4 (3)	1.00	1.00	1.00	16
V4-1 (4)	0.89	1.00	0.94	17
{e} (6)	1.00	1.00	1.00	179,910
Accuracy			1.00	180,198
Macro avg	0.98	0.98	0.98	180,198
Weighted avg	1.00	1.00	1.00	180,198

TABLE 6. Performance metrics using invariants as input features with class weighting.

The recall for minority classes improves dramatically: $C2$–1 increases from 0.10 to 0.93, $C2$–2 from 0.90 to 0.95, and the strata A_4, D_4, and $V4$–1 achieve perfect recall. The slight precision drop for $V4$–1 (to 0.89) arises from a small number of false positives and is consistent with the small sample size.

It is important to note that perfect performance on A_4 and D_4 is not a reflection of model sophistication: each of these loci consists of a single conjugacy class in $\mathcal{M}_3^1$, uniquely determined by its invariants. Similarly, high performance on $C2$–2 does not arise from a single criterion such as the vanishing of ξ_5, since the absolute invariants depend on all weighted coordinates (Eq. (25)) and uniquely determine the moduli point.

8. Conclusions and Further Directions

A central motivation for this work is the structural analogy between the moduli space of rational functions and the classical moduli spaces of hyperelliptic and superelliptic curves. In both settings, the moduli problem is governed by the action of PGL_2 on binary forms (or on suitable tuples of forms), the resulting quotients admit coordinate descriptions in terms of weighted projective invariants, and the loci where the objects acquire extra automorphisms are cut out by explicit relations among these invariants. For hyperelliptic curves this viewpoint underlies the rich invariant theory of binary forms; in the degree-three dynamical setting, an analogous picture arises for the pair $(F, G) \in V_4 \oplus V_2$, whose invariants $\xi_0, \ldots, \xi_5$ provide weighted-projective coordinates on $\mathcal{M}_3^1$ and, via the absolute invariants, determine the conjugacy class.

Within this framework, we identify $\mathcal{M}_3^1$ with the weighted projective space $\mathbb{P}^5_\omega$ (weights $(2, 2, 3, 3, 4, 6)$), give explicit generators $\xi_0, \ldots, \xi_5$ for the invariant map, and characterize the loci $\mathcal{L}_1, \ldots, \mathcal{L}_7$ corresponding to the finite subgroups C_2, C_3, C_4, V_4, D_4, and A_4 of PGL_2.

The construction of the dataset $\mathcal{P}_3^4$ further illustrates the arithmetic distribution of these loci inside $\mathcal{M}_3^1$, with the trivial automorphism locus dominating for small heights, just as in the hyperelliptic case where curves with extra automorphisms form thin subsets. This reinforces the geometric picture: enhanced symmetry is exceptional and detectable through the vanishing patterns of weighted invariants.

Section 7 provides only a brief computational illustration. No mathematical result of the paper depends on these experiments; they merely show that the invariants offer a natural feature representation for algorithmic classification and that the stratification of $\mathcal{M}_3^1$ is reflective in the numerical data.

Several avenues for further research arise naturally. The analogy with hyperelliptic moduli suggests studying higher-degree rational maps via invariant theory for $V_{d+1} \oplus V_{d-1}$, where the structure of the ring of invariants remains largely unexplored. Extending the explicit description of automorphism loci to larger d, constructing datasets of higher height, and developing symbolic or neuro-symbolic methods for approximating invariants in degrees where $\mathcal{R}_{(d+1,d-1)}$ is not yet known are promising directions.

In summary, this paper establishes a complete invariant–theoretic description of the moduli space $\mathcal{M}_3^1$, identifies all automorphism loci explicitly, and demonstrates how the geometry of these loci parallels classical phenomena from the theory of hyperelliptic and superelliptic curves. This provides both a conceptual bridge between two active areas of arithmetic geometry and a foundation for further investigations in arithmetic dynamics and invariant theory.

References

[1] E. Badr, E. Shaska, and T. Shaska, *Rational functions on the projective line from a computational viewpoint* (2026), available at 2503.10835.

[2] L. Beshaj, J. Gutierrez, and T. Shaska, *Weighted greatest common divisors and weighted heights*, J. Number Theory **213** (2020), 319–346. MR4091944

[3] L. Beshaj, R. Hidalgo, S. Kruk, A. Malmendier, S. Quispe, and T. Shaska, *Rational points in the moduli space of genus two*, Higher genus curves in mathematical physics and arithmetic geometry, 2018, pp. 83–115. MR3782461

[4] E. Curri, *On the stability of binary forms and their weighted heights*, Albanian J. Math. **16** (2022), no. 1, 3–23. MR4448533

[5] J. A. de Faria and B. Hutz, *Automorphism groups and invariant theory on* $\mathbb{P}^N$, J. Algebra Appl. **17** (2018), no. 9, 1850162, 38. MR3846410

[6] J.-C. Faugère, M. Safey El Din, and T. Verron, *On the complexity of computing Gröbner bases for weighted homogeneous systems*, J. Symbolic Comput. **76** (2016), 107–141. MR3461262

[7] B. Gontmacher, B. Hutz, G. Jorgenson, S. Srimani, and S. Xu, *Automorphism loci for degree 3 and degree 4 endomorphisms of the projective line*, New York J. Math. **27** (2021), 1613–1702. MR4359208

[8] J. Gutierrez and T. Shaska, *Hyperelliptic curves with extra involutions*, LMS J. Comput. Math. **8** (2005), 102–115. MR2135032

[9] V. Krishnamoorthy, T. Shaska, and H. Völklein, *Invariants of binary forms*, Progress in Galois theory, 2005, pp. 101–122. MR2148462

[10] N. Miasnikov, B. Stout, and P. Williams, *Automorphism loci for the moduli space of rational maps*, Acta Arith. **180** (2017), no. 3, 267–296. MR3709645

[11] S. Salami and T. Shaska, *Local and global heights on weighted projective varieties*, Houston J. Math. **49** (2023), no. 3, 603–636. MR4845203

[12] E. Shaska and T. Shaska, *Machine learning for moduli space of genus two curves and an application to isogeny-based cryptography*, J. Algebraic Combin. **61** (2025), no. 2, 23. MR4870337

[13] T. Shaska, *Determining the automorphism group of a hyperelliptic curve*, Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, 2003, pp. 248–254. MR2035219

[14] ———, *Subvarieties of the hyperelliptic moduli determined by group actions*, Serdica Math. J. **32** (2006), no. 4, 355–374. MR2287373

[15] T. Shaska and H. Völklein, *Elliptic subfields and automorphisms of genus 2 function fields*, Algebra, arithmetic and geometry with applications. Papers from Shreeram S. Ahhyankar's 70th birthday conference., 2003, pp. 703–723.

[16] T. Shaska, *Graded Neural Networks*, International Journal of Data Science in the Mathematical Sciences (2025), available at 2502.17751.

[17] ———, *A computational toolkit for degree 3 rational functions* (2026March). https://github.com/sha-aims/deg3-ratfuncs-moduli.

[18] J. H. Silverman, *Moduli spaces and arithmetic dynamics*, CRM Monograph Series, vol. 30, American Mathematical Society, Providence, RI, 2012. MR2884382

[19] L. W. West, *The moduli space of rational maps*, Ph.D. Thesis, 2015.

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