# GRADED QUANTUM CODES

T. SHASKA

*Department of Computer Science and Engineering,*
*Oakland University,*
*Rochester, MI, 48309*

Abstract. We introduce graded quantum codes, unifying two classes of quantum error-correcting codes. The first, quantum weighted algebraic geometry (AG) codes, derives from rational points on hypersurfaces in weighted projective spaces over finite fields. This extends classical AG codes by adding weighted degrees and singularities, enabling self-orthogonal codes via the CSS method with improved distances using algebraic structures and invariants like weighted heights.

The second class arises from chain complexes of graded vector spaces, generalizing homological quantum codes to include torsion and multiple gradings. This produces low-density parity-check codes with parameters based on homology ranks, including examples from knot invariants and quantum rotors.

A shared grading leads to a refined Singleton bound: $d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, where $\epsilon > 0$ reflects entropy adjustments from geometric singularities and defects. The bound holds partially for simple orbifolds and is supported by examples over small fields.

Applications include post-quantum cryptography, fault-tolerant quantum computing, and optimization via graded neural networks, linking algebraic geometry, homological algebra, and quantum information.

## 1. Introduction

Algebraic geometry (AG) codes, introduced by Goppa in the 1970s [19], form a foundational element of modern coding theory. These codes utilize rational points on algebraic varieties over finite fields to attain rates and distances surpassing those of classical constructions such as Reed–Solomon codes; see [48]. With the rise of quantum computing and vulnerabilities like Shor's algorithm as in [42], the need for quantum-resistant primitives has intensified, establishing code-based cryptography (e.g., the McEliece cryptosystem; see [27]) as a promising post-quantum approach. Quantum error correction safeguards delicate quantum states from decoherence, employing stabilizer codes (see [20]) and the Calderbank–Shor–Steane (CSS) framework in [11, 45] to derive codes from self-orthogonal classical counterparts.

This paper advances the field by introducing two interconnected classes of quantum error-correcting codes within the unifying paradigm of *graded quantum codes*.

2020 *Mathematics Subject Classification.* 94B05, 14G50, 55N30, 81P45.

*Key words and phrases.* Graded Quantum Codes, quantum weighted AG codes, homological quantum codes, weighted projective spaces, refined quantum Singleton bound.

The first class, *Quantum Weighted Algebraic Geometry Codes*, broadens classical AG codes to hypersurfaces in weighted projective spaces, integrating nontrivial gradings and orbifold singularities to produce self-orthogonal codes with superior parameters. These codes harness weighted homogeneous polynomials and arithmetic invariants, including weighted heights in [6, 31], for precise point counts and minimum distance bounds, frequently outperforming standard AG codes in rate and distance through semigroup structures and zeta functions; see [5, 16, 36, 37, 39, 40]

Complementing this, the second class, *Graded Quantum Codes*, emerges from chain complexes of graded vector spaces, extending homological quantum codes in [8, 47] to incorporate torsion modules and bigradings. This framework generates low-density parity-check (LDPC) codes applicable to fault-tolerant quantum computing, drawing on examples such as quantum rotor codes in [21] and Khovanov homology codes in [4].

A pivotal innovation is the shared grading mechanism linking the two classes, facilitating a refined Singleton-type bound: $d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, where $\epsilon > 0$ accommodates entropy corrections from orbifold cohomology in [12, 13] and topological defects. This bound is partially substantiated for $\mathbb{FP}^1$-orbifolds and corroborated by explicit examples over small finite fields. We further examine applications to post-quantum cryptography, including parameter optimization via graded neural networks in [37], and illustrate how these codes integrate algebraic geometry, homological algebra, and quantum information theory.

The paper is structured as follows. In Section 2 we introduce weighted projective spaces as a generalization of ordinary projective spaces, incorporating flexible gradings that account for orbifold singularities and toric structures, defined as quotients $(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$ under weighted actions with well-formed weights $\mathbf{w} = (w_0, \ldots, w_n)$ where $\gcd(w_i) = 1$, represented by orbits $[x_0 : \cdots : x_n]$ and covered by affine charts isomorphic to $\mathbb{A}^n/\mu_{w_i}$. Their coordinate ring is the graded algebra $S = \mathbb{F}[x_0, \ldots, x_n]$ with $\deg x_i = w_i$, where homogeneous ideals generate weighted varieties, such as hypersurfaces defined by polynomials $f \in S_d$ satisfying $f(\lambda^{w_i} x_i) = \lambda^d f(x_i)$. Rational points on these spaces over finite fields $\mathbb{F}_q$ are counted via orbit stratification, yielding $N_q = \sum_{\emptyset \neq S \subseteq [n]} (q-1)^{|S|-1} \gcd(k_S, q-1)$ where $k_S = \gcd(w_i : i \in S)$, with similar formulas for hypersurfaces restricting counts by zero loci. Zeta functions $Z_X(T)$ encode extension point counts $N_r$, satisfying Weil-like bounds $N_q \leq q^{\dim X} + O(q^{\dim X - 1/2})$ under smoothness, crucial for code lengths, while toric resolutions link to combinatorial fans for efficient computations in post-quantum applications.

In Section 3, we lay the foundation for quantum error-correcting codes by introducing classical linear and algebraic geometry (AG) codes, including weighted projective Reed-Muller (WPRM) and weighted AG codes on hypersurfaces in $\mathbb{WP}_{\mathbf{w}}$ over $\mathbb{F}_q$, with parameters derived from point counts, footprint bounds, and weighted heights; see [2, 16]. These are lifted to quantum stabilizer codes via the CSS construction, leveraging self-orthogonality for enhanced performance; see [11, 45]. We also explore homological codes, including toric and rotor variants, where gradings (e.g., torsion, bigradings) yield low-density parity-check (LDPC) codes with superior rates and distances; see [21, 24]. The toric perspective and height-based analysis unify these constructions, setting the stage for quantum extensions and refined bounds in subsequent sections.

In Section 4, we extend the classical code constructions from Section 3 to quantum weighted algebraic geometry (AG) codes, utilizing the CSS framework to create stabilizer codes from self-orthogonal weighted AG codes defined on hypersurfaces in $\mathbb{WP}_{\mathbf{w}}$ over $\mathbb{F}_q$; see [11, 45]. Self-orthogonality, achieved via geometric divisors or cyclic actions, ensures commuting stabilizers, yielding quantum codes with parameters $[[m, m-2k, d]]$ where $m = |X(\mathbb{F}_q)|$, $k = \dim C$, and $d \geq \min(d_C, d_{C^\perp})$; see [16, 38]. Examples include a hypersurface in $\mathbb{WP}_{(2,4,6,10)}$ over $\mathbb{F}_5$ producing $[[112, 92, d]]$, and a genus 2 curve over $\mathbb{F}_9$ yielding $[[8, 2, \geq 5]]$. Orbifold singularities contribute additional entropy via twisted sectors, motivating a refined Singleton bound $d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, where $\epsilon$ reflects cohomological corrections, enhancing fault-tolerance and cryptographic applications; see [1, 13].

In Section 5, we propose a refined quantum Singleton bound $d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, where $\epsilon > 0$ accounts for entropy corrections from orbifold singularities in weighted projective varieties, motivated by twisted sectors in Chen-Ruan cohomology and topological defects in TQFT; see [13, 15]. Supported by examples like Castle codes $[[10, 2, 3]]$ and toric LDPC codes $[[64, 16, 10]]$ over $\mathbb{F}_7$, and superelliptic curves with cyclic automorphisms yielding $[[12, 6, > 7]]$, the bound leverages geometric structures to enhance error correction; see [18, 38]. For $\mathbb{FP}^1$-orbifolds, partial validation uses explicit point counts and residue conditions, while homological and TQFT frameworks further justify the correction through spectral sequences and defect entropy, connecting algebraic geometry to quantum coding; see [4, 12].

In Section 6, we introduce graded quantum codes constructed from chain complexes of graded vector spaces over $\mathbb{F}_q$, generalizing homological quantum codes to produce low-density parity-check (LDPC) codes for fault-tolerant quantum computing; see [8]. These codes leverage gradings (e.g., integer-valued, torsion, or bigraded) to define stabilizers via differentials, with parameters $[[n, k, d]]$ where $n = \dim C_d$, $k = \dim H_d(C_\bullet)$, and $d$ satisfies a refined Singleton bound adjusted by orbifold corrections $\varepsilon = \frac{1}{2}\sum_p (1 - \frac{1}{|G_p|})$; see [15]. By integrating weighted AG codes from Section 3 using height zeta functions and vanishing ideals, we construct scalable codes, exemplified by Khovanov homology codes $[[10, 3, 4]]$ for torus knots and height-graded codes $[[10, 2, 3]]$ on superelliptic curves; see [4, 16]. Finally, Section 7 addresses applications and prospective directions.

## 2. Preliminaries

2.1. **Weighted Projective Spaces.** Weighted projective spaces are a natural generalization of ordinary projective spaces, allowing for more flexible gradings that capture orbifold singularities and toric structures. They arise as quotients by weighted actions and are particularly useful in algebraic geometry over finite fields for counting rational points and constructing codes.

**Definition 2.1.** Let $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ be positive integers, called *weights*, with $\gcd(w_0, \ldots, w_n) = 1$ (well-formed condition). The *weighted projective space* $\mathbb{WP}_{\mathbf{w}}$ over a field $\mathbb{F}$ is the quotient scheme $(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$, where the multiplicative group $\mathbb{G}_m = \mathbb{F}^*$ acts by $\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{w_0} x_0, \ldots, \lambda^{w_n} x_n)$.

This space can be viewed as a geometric invariant theory (GIT) quotient or as a stack, with points represented by orbits $[x_0 : \cdots : x_n]$. It is covered by affine charts $U_i = \{x_i \neq 0\}$, isomorphic to $\mathbb{A}^n/\mu_{w_i}$, where $\mu_{w_i}$ is the group of $w_i$-th roots of unity, introducing orbifold singularities unless $w_i = 1$.

The *coordinate ring* is the graded $\mathbb{F}$-algebra $S = \mathbb{F}[x_0, \ldots, x_n]$ with $\deg x_i = w_i$, so $S = \bigoplus_{d \geq 0} S_d$, where $S_d$ is the span of monomials of weighted degree $d = \sum w_i e_i$. Ideals in $S$ are weighted homogeneous if generated by weighted homogeneous elements.

**Example 2.2.** Hypersurfaces in $\mathbb{WP}_{\mathbf{w}}$ are defined by weighted homogeneous polynomials $f \in S_d$ for some degree $d$. For example, in $\mathbb{WP}_{(1,1,2)}$, a quartic hypersurface might be given by $x_2^2 - x_0^4 + x_1^4 = 0$, which is weighted homogeneous of degree 4.

Over finite fields $\mathbb{F}_q$, the set of rational points $\mathbb{WP}_{\mathbf{w}}(\mathbb{F}_q)$ consists of orbits with representatives where not all coordinates are zero, normalized appropriately. The number of such points is given by

$$N_q = \sum_{\emptyset \neq S \subseteq [n]} (q-1)^{|S|-1} \gcd(k_S, q-1),$$

where $k_S = \gcd(w_i : i \in S)$, and the sum is over nonempty subsets $S$. This accounts for all orbits, and for well-formed spaces, no further adjustments are needed.

The *zeta function* of a hypersurface $X \subset \mathbb{WP}_{\mathbf{w}}$ over $\mathbb{F}_q$ is

$$Z_X(T) = \exp\left( \sum_{r=1}^{\infty} \frac{N_r T^r}{r} \right),$$

where $N_r = \#X(\mathbb{F}_{q^r})$ is the number of points over the extension $\mathbb{F}_{q^r}$. For weighted hypersurfaces, analogs of the Weil conjectures hold under smoothness assumptions, providing bounds like

$$N_q \leq q^{\dim X} + O(q^{\dim X - 1/2}),$$

see [2] for details. These bounds are crucial for estimating code lengths in weighted algebraic geometry codes. Recent developments enable explicit computations of zeta functions for certain hypersurfaces, yielding point counts over small fields and applications to post-quantum cryptography; see [28, 28].

Weighted projective spaces often admit toric resolutions, linking them to combinatorial geometry via fans and polytopes, which aids in computing point counts and code parameters.

2.2. **Weighted Varieties.** A *weighted variety* is a subvariety of a weighted projective space, defined by *weighted homogeneous polynomials*. A polynomial $f(x_0, x_1, \ldots, x_n)$ is *weighted homogeneous of degree $d$* with respect to the weights $\mathbf{w}$ if:

$$f(\lambda^{w_0} x_0, \lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^d f(x_0, x_1, \ldots, x_n) \quad \text{for all } \lambda \in \mathbb{F}^*.$$

This condition ensures that the zero set of $f$ is well-defined in $\mathbb{WP}_{\mathbf{w}}$.

A *weighted hypersurface* $X \subset \mathbb{WP}_{\mathbf{w}}$ is defined as the zero set of a weighted homogeneous polynomial $f$:

$$X = \{[x_0 : x_1 : \cdots : x_n]_{\mathbf{w}} \in \mathbb{WP}_{\mathbf{w}} \mid f(x_0, x_1, \ldots, x_n) = 0\}.$$

*Remark* 2.3. The weight grading on $S$ is independent of the homological grading in chain complexes; they combine in graded quantum codes via total degree or filtered subcomplexes (see Section 6)."

2.3. **Rational Points on Weighted Varieties.** For a variety $X$ defined over a field $\mathbb{F}$, the $\mathbb{F}$-*rational points* (or simply rational points) are the points on $X$ with coordinates in $\mathbb{F}$, up to the equivalence defining the space. Specifically, for a weighted variety $X \subset \mathbb{WP}_{\mathbf{w}}$, the set of rational points $X(\mathbb{F})$ consists of equivalence classes $[x_0 : x_1 : \cdots : x_n]_{\mathbf{w}}$ where $x_i \in \mathbb{F}$, not all zero, and $f(x_0, x_1, \ldots, x_n) = 0$.

When $\mathbb{F} = \mathbb{F}_q$ is a finite field with $q$ elements, we are often interested in counting the number of $\mathbb{F}_q$-rational points, denoted $|X(\mathbb{F}_q)|$.

2.3.1. *Counting Rational Points on Weighted Projective Spaces.* To count the rational points on $\mathbb{WP}_{\mathbf{w}}$ over $\mathbb{F}_q$, we consider the action of $\mathbb{F}_q^*$ on $\mathbb{A}^{n+1}(\mathbb{F}_q) \setminus \{0\}$. Each point $(x_0, x_1, \ldots, x_n) \in \mathbb{A}^{n+1}(\mathbb{F}_q) \setminus \{0\}$ lies in an orbit under the weighted scaling, and the rational points correspond to these orbits.

The size of an orbit depends on the *stabilizer* of a point. For a point with support $S = \{i \mid x_i \neq 0\}$, the stabilizer consists of $\lambda \in \mathbb{F}_q^*$ such that $\lambda^{w_i} = 1$ for all $i \in S$, i.e., $\lambda^k = 1$ where $k = \gcd(\{w_i \mid i \in S\})$. Thus, the stabilizer has order $\gcd(k, q-1)$, and the orbit size is:

$$\frac{|\mathbb{F}_q^*|}{|\text{stabilizer}|} = \frac{q-1}{\gcd(k, q-1)}.$$

Let $N_S$ be the number of points in $\mathbb{A}^{n+1}(\mathbb{F}_q) \setminus \{0\}$ with $x_i \neq 0$ for $i \in S$ and $x_i = 0$ for $i \notin S$. Notice that

$$N_S = (q-1)^{|S|},$$

where $|S|$ is the cardinality of $S$. The number of rational points contributed by these points is:

$$\frac{N_S \cdot \gcd(k_S, q-1)}{q-1}, \quad \text{where } k_S = \gcd(\{w_i \mid i \in S\}).$$

Summing over all nonempty subsets $S \subseteq \{0, 1, \ldots, n\}$, the total number of rational points on $\mathbb{WP}_{\mathbf{w}}$ is

$$|\mathbb{WP}_{\mathbf{w}}(\mathbb{F}_q)| = \sum_{S \neq \emptyset} \frac{N_S \cdot \gcd(k_S, q-1)}{q-1} = \sum_{S \neq \emptyset} (q-1)^{|S|-1} \gcd(k_S, q-1). \tag{1}$$

2.4. **Counting Rational Points on Weighted Varieties.** For a weighted variety $X \subset \mathbb{WP}_{\mathbf{w}}$ defined by a weighted homogeneous polynomial $f = 0$ of degree $d$, the rational points are the orbits in:

$$\{(x_0, x_1, \ldots, x_n) \in \mathbb{A}^{n+1}(\mathbb{F}_q) \setminus \{0\} \mid f(x_0, x_1, \ldots, x_n) = 0\}.$$

The number of rational points can be similarly computed by stratifying based on the support $S$:

$$|X(\mathbb{F}_q)| = \sum_{S \neq \emptyset} \frac{N_S \cdot \gcd(k_S, q-1)}{q-1},$$

where $N_S$ now counts points with support $S$ such that $f(x_0, x_1, \ldots, x_n) = 0$.

2.5. **Bounds on Rational Points on Weighted Hypersurfaces.** In classical projective space, Serre's inequality provides an upper bound on the number of $\mathbb{F}_q$-rational points on a hypersurface. This was extended to weighted projective spaces by Aubry et al., leading to the conjecture:

$$|X(\mathbb{F}_q)| \leq q^{n-1} + \frac{d}{\min_i w_i} q^{n-2} + \cdots + \frac{d^{n-1}}{\prod_{i=1}^{n-1} \min w_{j_i}},$$

where $p_n = |\mathbb{P}^n(\mathbb{F}_q)|$.

The conjecture generalizes the Tsfasman-Bogomolov bound for unweighted hypersurfaces, where all weights are 1, giving $|X(\mathbb{F}_q)| \leq q^{n-1} + dq^{n-2} + \cdots + d^{n-1}$.

In the weighted case, assume without loss of generality that the weights $w_0, \ldots, w_n$ satisfy $\gcd(w_0, \ldots, w_n) = 1$ and are ordered $w_0 = 1 \leq w_1 \leq \cdots \leq w_n$ (relabeling coordinates if necessary, as the space is isomorphic under permutation).

The bound is derived inductively by slicing the hypersurface with weighted hyperplanes of minimal degree $\min w_i$, reducing the dimension at each step.

The term $\frac{d}{\min w_i}$ arises as the effective number of slices in the first induction step, maximizing the bound by choosing the smallest weight. Subsequent terms use the next minimal weights among the remaining coordinates.

This form is conjectured to hold when $\operatorname{lcm}(w_1, \ldots, w_n) \mid d$ and $d \leq w_1(q+1)$ (to avoid saturation at $p_n$); see [2] (Conjecture 1, pg. 4) for the initial formulation and [3] for refinements removing the divisibility condition, replacing $\frac{d}{w_1}$ with $\left\lfloor \frac{d-1}{w_1} \right\rfloor + 1$. In [2] (see Theorem$\tilde{1}$, pg. 9) it was proved that for certain weighted homogeneous polynomials,

$$|X(\mathbb{F}_q)| \leq q^{n-1} + \frac{d}{w_1} q^{n-2} + p_{n-3},$$

assuming the second smallest weight is 1. This bound applies under specific conditions, particularly when there exists a hyperplane $H$ such that $|X(\mathbb{F}_q) \cap H| = 0$. More precisely, in [3], Theorem 5.1, p. 10 proves the bound when $w_1 = 1$ (second smallest weight after $w_0 = 1$):

$$e_q(d; 1, 1, w_2, \ldots, w_n) = \min\left\{p_n, dq^{n-1} + p_{n-2}\right\},$$

where $e_q(d; w)$ denotes the maximum number of $\mathbb{F}_q$-points over all degree-$d$ hypersurfaces. Here $p_{n-2} = q^{n-2} + q^{n-3} + \cdots + 1$, but if the next weight $w_2 > 1$, the induction approximates with $p_{n-3}$ in lower terms due to orbifold effects. The proof combines an upper bound via Gröbner bases and footprint techniques with a lower bound by explicit construction.

For the upper bound: Consider the vanishing ideal $I$ of the $\mathbb{F}_q$-points of $\mathbb{P}_{\mathbf{w}}$, where $\mathbf{w} = (1, 1, w_2, \ldots, w_n)$. Using a weighted degree-reverse-lexicographic monomial order, the initial ideal $\operatorname{in}(I)$ is generated by certain monomials.

For a nonzero $f$ of degree $d$, the number of $\mathbb{F}_q$-points on $V(f)$ is at most $p_n - |\Delta(\operatorname{in}(f))|$, where $\Delta(m)$ is the footprint (monomials not divisible by the leading monomial $m = \operatorname{in}(f)$). Minimizing the footprint over possible $m$ of degree $d$ yields the bound $dq^{n-1} + p_{n-2}$.

For the lower bound: Construct $F$ as a product of $d$ weighted linear factors (possible since weights include 1's), vanishing on exactly $dq^{n-1} + p_{n-2}$ points, as the zeros cover $d$ hyperplanes in affine charts.

Recent extensions in [10] prove the conjecture for hypersurfaces in three-dimensional weighted projective spaces, and [33] Theorem 1.2, p. 2 refines it further.

In [34] Serre established that for a weighted homogeneous polynomial $F$ in $\mathbb{P}(w_0, \ldots, w_n)$ with $d \leq n$:

$$|X(\mathbb{F}_q)| \equiv 1 \pmod{p},$$

where $p$ is the characteristic of $\mathbb{F}_q$. A stronger conjecture (Aubry et al. [2]) suggests that $|X(\mathbb{F}_q)| \equiv 1 \pmod{q}$. Serre's congruence generalizes from the unweighted case using the Chevalley-Warning theorem on the affine cone: the number of solutions to

$F = 0$ in $\mathbb{A}^{n+1} \setminus \{0\}$ is congruent to 0 modulo $p$ if the sum of degrees is less than the dimension, then quotioning by the scalar action gives $\equiv 1 \pmod{p}$. The weighted version requires adjusting for weighted homogeneity, as shown in [26]. The mod $q$ conjecture holds partially for unirational varieties (see [30]) and specific weighted cases where $|X(\mathbb{F}_{q^k})| \equiv 1 + q^k + \cdots + q^{k(\mu-1)} \pmod{q^{k\mu}}$ for $\mu \geq 1$, implying $\equiv 1 \pmod{q}$ when higher terms vanish.

## 3. Code Constructions

To set the stage for our main constructions, we first recall the foundational notions of classical linear codes and their geometric variants, which provide the building blocks for quantum error-correcting codes. These classical codes, particularly those arising from algebraic varieties, offer rich parameters that can be lifted to the quantum setting through techniques like the Calderbank–Shor–Steane (CSS) construction. We then transition to quantum stabilizer codes and homological frameworks, emphasizing how gradings enrich these structures to yield codes with enhanced properties, such as low-density parity-check (LDPC) behavior.

### 3.1. Classical Linear and Geometric Codes.

Consider a finite field $\mathbb{F}_q$ of characteristic $p$, and let $V = \mathbb{F}_q^n$ denote the vector space of length-$n$ words over $\mathbb{F}_q$.

**Definition 3.1.** A *linear code* $\mathcal{C} \subseteq \mathbb{F}_q^n$ is a $k$-dimensional subspace of $V$. The *minimum distance* $d$ is the smallest Hamming weight (number of nonzero entries) among its nonzero codewords. Such a code has parameters $[n, k, d]_q$.

Linear codes detect up to $d-1$ errors and correct up to $\lfloor (d-1)/2 \rfloor$ errors, making the parameters $n$, $k$, and $d$ central to their performance.

A powerful method to construct linear codes involves evaluating functions on algebraic varieties.

**Definition 3.2.** Let $X \subseteq \mathbb{A}^m$ be an algebraic variety over $\mathbb{F}_q$, and let $\mathcal{L} \subset \mathbb{F}_q[X]$ be a finite-dimensional subspace of regular functions on $X$. Given a set of $n$ distinct rational points $P = \{P_1, \ldots, P_n\} \subset X(\mathbb{F}_q)$, the associated *evaluation code* is the subspace

$$\mathcal{C}(X, \mathcal{L}, P) := \{(f(P_1), \ldots, f(P_n)) \mid f \in \mathcal{L}\} \subset \mathbb{F}_q^n.$$

The dimension of this code is at most $\dim \mathcal{L}$, with equality if the evaluation map is injective (i.e., no nonzero function in $\mathcal{L}$ vanishes on all of $P$). The minimum distance is bounded below by $n$ minus the maximum number of zeros that a nonzero function in $\mathcal{L}$ has on $P$, provided the evaluation map is injective. In weighted projective spaces, zero-counting respects orbit stabilizers under the weighted action (e.g., via $\gcd(k_S, q-1)$ for subsets $S$ of coordinates).

When $X = \mathbb{P}^m$ and $\mathcal{L} = H^0(\mathbb{P}^m, \mathcal{O}(d))$, the space of homogeneous polynomials of degree $d$, the resulting codes are known as *projective Reed–Muller codes*. These generalize Reed–Solomon codes (the case $m = 1$) and achieve good parameters for small $d$.

A particularly fruitful subclass arises when $X$ is a smooth projective curve.

**Definition 3.3.** Let $C$ be a smooth projective curve over $\mathbb{F}_q$ of genus $g$, let $\mathcal{P} = \{P_1, \ldots, P_n\} \subset C(\mathbb{F}_q)$ be a set of distinct rational points, and let $G$ be a divisor on $C$ with support disjoint from $\mathcal{P}$. The *algebraic geometry (AG) code* is

$$\mathcal{C}_L(C, G, \mathcal{P}) := \{(f(P_1), \ldots, f(P_n)) \mid f \in L(G)\},$$

where $L(G) = \{f \in \mathbb{F}_q(C) : \div(f) + G \geq 0\} \cup \{0\}$ is the Riemann–Roch space associated to $G$.

By the Riemann–Roch theorem, $\dim L(G) \geq \deg G + 1 - g$, with equality if $\deg G > 2g - 2$. AG codes often surpass the classical Gilbert–Varshamov bound and include many record-holding constructions; see [48].

3.2. **Weighted Projective Reed-Muller Codes.** Let $\mathbb{F}_q$ be a finite field with $q$ elements, and let $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ be a tuple of positive integers with $\gcd(w_0, \ldots, w_n) = 1$.

Weighted Projective Reed-Muller Codes were introduced in [29, 44] and are defined as follows:

**Definition 3.4.** Let $\mathcal{P} = \{P_1, \ldots, P_m\} = \mathbb{WP}_{\mathbf{w}}(\mathbb{F}_q)$. The *WPRM code* $\mathrm{WPRM}_d(\mathbf{w})$ is the image of the evaluation map $\mathrm{ev}_d : S_d \to \mathbb{F}_q^m$, such that

$$f \mapsto (f(P_1), \ldots, f(P_m)).$$

The kernel is $I(\mathcal{P})_d = S_d \cap I(\mathcal{P})$, where $I(\mathcal{P})$ is the weighted homogeneous vanishing ideal. The length is $m = |\mathbb{WP}_{\mathbf{w}}(\mathbb{F}_q)|$, computed as:

$$m = \sum_{S \neq \emptyset} (q-1)^{|S|-1} \gcd(k_S, q-1),$$

where $k_S = \gcd(\{w_i \mid i \in S\})$. For planes $\mathbb{WP}_{(1,w_1,w_2)}$ with $\gcd(w_1, w_2) = 1$, explicit counting formulas apply.

3.3. **Weighted AG Codes from Hypersurfaces.** A *weighted hypersurface* $X \subset \mathbb{WP}_{\mathbf{w}}$ is defined by a weighted homogeneous polynomial $f \in S_e$ of degree $e$. Let $\mathcal{P} = X(\mathbb{F}_q) = \{P_1, \ldots, P_m\}$.

**Definition 3.5** (Weighted AG Code)**.** The *weighted AG code* $C(X, d)$ is the image of $\mathrm{ev}_d : S_d \to \mathbb{F}_q^m$.

The length $m = |X(\mathbb{F}_q)|$ is bounded by generalized Serre inequalities (see [2]) as follows:

$$m \leq \min\left\{ p_{n-1}, \frac{e}{w_1} q^{n-2} + p_{n-3} \right\},$$

where $p_k = |\mathbb{P}^k(\mathbb{F}_q)|$ and weights are ordered $w_1 \leq \cdots \leq w_{n-1}$, assuming the second-smallest weight is 1. The zeta function $Z_X(T)$ provides point counts over extensions, satisfying Weil-type bounds under smoothness.

3.4. **Minimum Distance and Dimension Bounds.** The dimension $k = \dim S_d - \dim I(\mathcal{P})_d$, where $\dim S_d = \mathrm{den}(d; \mathbf{w})$, the number of non-negative integer solutions to $\sum w_i a_i = d$.

For WPRM on planes:

**Theorem 3.6** (Dimension for WPRM on Planes; see [29])**.** *Let* $\mu_1 = \min(\lfloor d/w_1 \rfloor, q-1)$, $\mu_2 = \min(\lfloor d/w_2 \rfloor, q-1)$, $\ell = \max(0, \min(q-1, \lfloor (d - w_1(q-1))/w_2 \rfloor))$. *Then*

$$k = \sum_{i=0}^{\mu_1} (\lfloor (d - w_1 i)/w_2 \rfloor + 1) - \sum_{i=0}^{\ell} (q - 1 - i) + \textit{corrections for overlaps.}$$

*Proof.* Count independent monomials, subtracting redundancies from the vanishing ideal's Gröbner basis. □

For hypersurfaces, use the Hilbert function of $S/(I(X) + I(\mathcal{P}))$. The minimum distance $d(C) = m - \max\{|Z(()f)| \mid f \in S_d \setminus \{0\}\}$.

**Theorem 3.7** (Footprint Bound for Minimum Distance)**.** *For the WPRM code $C_d$ on $\mathbb{WP}_{\mathbf{w}}$, the minimum distance satisfies*

$$d(C) = n - \max\{|Z(f)| : f \in S_d \setminus \{0\}\},$$

*where the footprint $\Delta(f)$ provides the bound, adapted to the weighted graded ring: Restrict monomials to those of weighted degree $d = \sum w_i e_i$, and adjust zero-counting for orbit stabilizers via $\gcd(k_S, q-1)$ in inclusion-exclusion over subsets $S$.*

*Proof.* The footprint $\Delta(f)$ is the set of monomials in the weighted graded component $S_d$ not divisible by leading terms of the Groebner basis of $I(Y) + (f)$. The bound follows as $|Z(f)| \leq |\Delta(f)|$, with equality if the basis is monomial. In the weighted setting, monomials are filtered by degree $d$, and gcd constraints adjust orbit multiplicities in zero loci: For $f$ vanishing on an orbit $S$, contribute $|\operatorname{orb}| = (q-1)/\gcd(k_S, q-1)$ to the count. □

In the case of weighted projective planes ($n = 2$), the footprint method yields the exact minimum distance when the Gröebner basis is monomial and orbit stabilizers are incorporated:

$$d(C) = n - \max_f \#\Delta(f),$$

where $\Delta(f)$ is the set of weighted monomials of degree $d$ not in the initial ideal of $I(Y) + (f)$, adjusted for gcd constraints on orbit lengths as in [10].

3.5. **Combinatorics and Duality.** The weight distribution links to hypersurface counts with fixed zeros, relating to designs and association schemes. The automorphism group includes the weighted projective linear group.

**Theorem 3.8.** *Let $C_d$ be the WPRM code on $\mathbb{WP}_{\mathbf{w}}$ of degree $d$, with parameters $[n, k, d_{\min}]$. The dual code $C_d^\perp$ corresponds to evaluation on the dual weighted space $\mathbb{WP}_{\mathbf{w}^*}$, where $\mathbf{w}^* = (w_n, \ldots, w_0)$ is the reversed weight vector, under the MacWilliams duality (orthogonal complement with respect to the Hermitian inner product when $q$ is a square).*

*Proof.* The duality is MacWilliams-type: For a code $C = \operatorname{ev}(S_d)$ over $\mathbb{F}_{q^2}$, the dual $C^\perp = \{\mathbf{c}' \in \mathbb{F}_{q^2}^n \mid \langle \mathbf{c}, \mathbf{c}' \rangle_H = 0 \ \forall \mathbf{c} \in C\}$, where $\langle \mathbf{u}, \mathbf{v} \rangle_H = \sum u_i \bar{v}_i$ and $\bar{v}_i = v_i^q$ (Frobenius conjugate). Reversed weights $\mathbf{w}^*$ ensure the dual monomials remain homogeneous of degree $d$ in the reversed grading, preserving the weighted structure. Hypotheses: Assume $q$ is square (for Hermitian product) and $C$ self-orthogonal (as in CSS constructions). The proof follows from Serre duality on the resolved toric variety: The canonical divisor adjusts degrees inversely, mapping $S_d$ to $S_{K-d}$ where $K$ is weighted dual degree, yielding reversed weights. Explicitly, the generator matrix of $C^\perp$ is orthogonal to that of $C$, and reversal maintains injectivity if original evaluation is. □

Connections to sphere packings via MacWilliams identity and Hamming bounds; weighted metrics align with subspace distances for constant-weight subcodes.

3.6. **Quantum Stabilizer Codes.** Quantum error-correcting codes protect fragile quantum information encoded in states of a Hilbert space $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$, the tensor product of $n$ $q$-dimensional qudit spaces.

**Definition 3.9.** The *generalized Pauli (or Weyl–Heisenberg) group* on $n$ qudits is the group generated by the shift operator $X$ and phase operator $Z$, satisfying the commutation relation $ZX = \omega XZ$, where $\omega = e^{2\pi i/q}$ is a primitive $q$-th root of unity; see [20]. A *stabilizer code* is defined by an abelian subgroup $\mathcal{S}$ of this group such that $-I \notin \mathcal{S}$, with the code space given by the simultaneous $+1$ eigenspace:

$$\mathcal{Q} := \{\psi \in \mathcal{H} : s\psi = \psi \ \forall s \in \mathcal{S}\}.$$

If $\mathcal{S}$ has $n-k$ independent generators, the code encodes $k$ logical qudits into $n$ physical qudits and has parameters $[[n, k, d]]$, where $d$ is the minimum weight of an undetectable error (an operator in the centralizer of $\mathcal{S}$ but not in $\mathcal{S}$ itself).

Stabilizer codes can detect errors that anticommute with at least one element of $\mathcal{S}$ or act trivially on the code space.

3.7. **CSS Codes and Quantum AG Codes.** A prominent method to construct stabilizer codes is the Calderbank–Shor–Steane (CSS) framework, which builds quantum codes from pairs of classical codes; see [11, 45] for details.

**Definition 3.10** (CSS Construction)**.** Let $C_1, C_2 \subseteq \mathbb{F}_q^n$ be linear codes such that $C_2^\perp \subseteq C_1$. The corresponding *CSS code* has parameters

$$[[n, \dim C_1 - \dim C_2, d]],$$

where $d = \min\{\mathrm{wt}(C_1 \setminus C_2^\perp), \mathrm{wt}(C_2 \setminus C_1^\perp)\}$.

The $X$-type stabilizers are derived from the parity-check matrix of $C_1$, while the $Z$-type stabilizers come from that of $C_2$. The orthogonality condition ensures that the stabilizers commute.

**Example 3.11** (Quantum AG Codes)**.** Suppose $C = C_L(C, G, \mathcal{P})$ is an AG code on a curve $C$ that satisfies $C \subseteq C^\perp$ (e.g., self-orthogonal under a suitable inner product, often achieved via residues or involutions on the curve; see [38]). Then the CSS construction yields a quantum code that can exceed the quantum Gilbert–Varshamov bound in certain cases.

The Tsfasman–Vladut–Zink bound guarantees that, over sufficiently large finite fields, families of curves exist for which AG codes produce asymptotically good quantum codes.

3.8. **Homological and LDPC Quantum Codes.** A complementary approach constructs quantum codes from the homology of topological or combinatorial structures.

**Definition 3.12.** A *chain complex* $C_\bullet$ over $\mathbb{F}_q$ is a sequence of vector spaces with differentials $\partial_i : C_i \to C_{i-1}$ satisfying $\partial_{i-1} \circ \partial_i = 0$. Given an inner product on the spaces, the adjoint differential $\partial_i^\dagger : C_{i-1} \to C_i$ satisfies $\langle \partial_i x, y\rangle = \langle x, \partial_i^\dagger y\rangle$. For a suitable degree $d$, define:

- $X$-type stabilizers as $\mathrm{im}(\partial_{d+1}) \subset C_d$,
- $Z$-type stabilizers as $\mathrm{im}(\partial_d^\dagger) \subset C_d$,

provided the self-orthogonality condition $\mathrm{im}(\partial_{d+1}) \subseteq \ker(\partial_d^\dagger)$ holds. The logical qubits correspond to the homology group $H_d(C_\bullet) = \ker(\partial_d)/\mathrm{im}(\partial_{d+1})$; see [8].

This generalizes Kitaev's surface code and often results in low-density parity-check (LDPC) codes when the differentials are sparse (i.e., each generator acts on a bounded number of sites).

**Example 3.13** (Toric Code)**.** The toric code arises from the cellular chain complex of a torus tiled by squares: $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, where $C_i$ is generated by $i$-cells. Stabilizers are defined at degree 1, yielding parameters $[[2L^2, 2, L]]$, where $L$ is the lattice size; see [24].

**Theorem 3.14** (Tillich–Zémor [47])**.** *There exist families of quantum LDPC codes from homological products of chain complexes with parameters*

$$[[n, k = \Theta(n), d = \Theta(\sqrt{n})]].$$

3.9. **Graded Extensions of Homological Codes.** Traditional homological quantum codes are built from ungraded or trivially graded chain complexes over finite fields. Recent innovations enrich these complexes with nontrivial gradings, torsion modules, or bigradings, leading to codes with superior parameters, such as higher rates, better distances, or enhanced fault tolerance.

One such class incorporates torsion to model continuous symmetries or defects.

**Definition 3.15** (Quantum Rotor Codes [21])**.** Quantum rotor codes are constructed from chain complexes of graded modules over rings with torsion, such as the circle group $\mathbb{T}$ or $U(1)$ symmetries. Consider a chain complex $C_\bullet$ of graded abelian groups or $R$-modules, where $R$ contains torsion elements. The grading refines the homological degree, and torsion submodules

$$H_d^{\mathrm{tors}}(C_\bullet) := \{x \in H_d(C_\bullet) \mid \exists n \in \mathbb{N},\ nx = 0\}$$

encode logical qubits via topological defects. Stabilizers are derived from boundaries and coboundaries, with additional logical operators from torsion classes.

**Example 3.16.** Rotor codes generalize Kitaev's toric code by replacing finite-dimensional qubits with infinite-dimensional rotor Hilbert spaces. The grading corresponds to spatial dimensions, while torsion encodes flux defects, yielding codes with improved error thresholds and continuous-symmetry logical operators.

Another extension uses bigraded complexes from link invariants.

**Definition 3.17** (Khovanov Homology Quantum Codes [4])**.** Khovanov homology assigns to a link $L$ a bigraded chain complex $(C^{i,j}(L), d)$ with homological grading $i$ and quantum grading $j$, constructed combinatorially from resolutions of a link diagram. Quantum codes are built by applying the CSS construction to subcomplexes filtered by the quantum grading:

$$C^{\leq k} = \bigoplus_{j \leq k} C^{*,j}.$$

The parameters depend on the ranks and torsion of the bigraded homology groups $H^{i,j}(L)$.

**Example 3.18.** For torus knots, Khovanov-based codes have parameters tied to the knot's topology, exploiting the bigrading to achieve minimum distances beyond classical homological bounds.

Finally, topological quantum field theories (TQFTs) with defects provide a graded framework for anyon-based codes.

**Definition 3.19** (TQFT-Based Codes with Defects)**.** These codes arise from graded unitary modular tensor categories or extended TQFTs modeling topological phases. Defects correspond to grading the category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ by a symmetry group $G$. The associated chain complex inherits the grading, with stabilizers defined on graded components indexed by defect types or charges.

**Example 3.20.** Graded TQFT models with finite-group defects yield codes with improved fault tolerance by localizing errors in graded sectors, often exhibiting higher thresholds than ungraded analogs.

In each case, the grading—whether integer-valued, multi-indexed, or group-graded—enriches the code's algebraic structure, manifesting in refined syndromes, enhanced symmetries, and superior parameters. This motivates our notion of *graded quantum codes*, which integrate gradings with arithmetic and geometric data, such as weighted point counts and heights, to control code properties systematically.

3.10. **Quantum Error Correction Basics.** Quantum information is inherently fragile due to interactions with the environment, leading to decoherence and errors that can corrupt computations. Unlike classical bits, quantum states cannot be directly copied (by the no-cloning theorem; see [50]), necessitating specialized error correction techniques. Quantum error correction (QEC) encodes logical qubits into multiple physical qubits, allowing detection and correction of errors without measuring the quantum state.

Common error models include:

(1) *Bit-flip error:* The Pauli $X$ operator, mapping $|0\rangle \to |1\rangle$ and $|1\rangle \to |0\rangle$.
(2) *Phase-flip error:* The Pauli $Z$ operator, mapping $|+\rangle \to |-\rangle$ while leaving the computational basis unchanged.
(3) *Combined error:* The Pauli $Y = iXZ$.
(4) *General errors:* Modeled as linear combinations of $I, X, Y, Z$ on each qubit.

The Pauli group on $n$ qubits is

$$\mathcal{P}_n = \{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n},$$

with group operation defined up to global phase.

Stabilizer codes, introduced by Gottesman; see [20], provide a powerful framework for QEC. They define the code space as the simultaneous $+1$ eigenspace of a set of commuting Pauli operators.

**Definition 3.21** (Stabilizer Group)**.** A stabilizer group $S$ is an abelian subgroup of $\mathcal{P}_n$ such that $-I \notin S$. The corresponding code space is the $+1$ eigenspace:

$$V_S = \left\{ |\psi\rangle \in (\mathbb{C}^2)^{\otimes n} : s\,|\psi\rangle = |\psi\rangle \ \ \forall s \in S \right\}.$$

If $S$ is generated by $n-k$ independent elements, the code encodes $k$ logical qubits into $n$ physical qubits and is denoted by $[[n, k]]$.

**Theorem 3.22** (Error Detection in Stabilizer Codes)**.** *An error $E$ (a Pauli operator) is detectable if either $E \in S$ (i.e., acts trivially on the code space), or $E$ anticommutes with at least one $s \in S$ (yielding a nonzero syndrome).*

*Proof.* Let $|\psi\rangle \in V_S$. If $E$ commutes with all $s \in S$ and $E \notin S$, then $E|\psi\rangle$ lies in a distinct eigenspace orthogonal to $V_S$. If $E$ anticommutes with some $s \in S$, then measuring $s$ yields eigenvalue $-1$, thus detecting the error. The code distance $d$ is the minimum weight of an undetectable error. □

To correct errors: we measure the stabilizer generators (syndromes), identify the error via a lookup table, and apply the inverse operator.

Calderbank–Shor–Steane (CSS) codes( as in [11, 45]) construct quantum codes from pairs of classical linear codes.

**Definition 3.23** (CSS Code)**.** Let $C_1, C_2$ be classical linear codes over $\mathbb{F}_2$ with parameters $[n, k_1, d_1]$ and $[n, k_2, d_2]$, respectively, such that $C_2^\perp \subseteq C_1$. Then the CSS code has parameters

$$[[n, k_1 + k_2 - n, d]], \quad d = \min\left(d(C_1 \setminus C_2^\perp),\ d(C_2 \setminus C_1^\perp)\right).$$

Stabilizer generators are defined as:

- $X$-type from the parity checks of $C_1$,
- $Z$-type from the parity checks of $C_2$.

**Theorem 3.24** (CSS Construction)**.** *The stabilizers commute, form an abelian group, and detect all errors of weight less than $d$.*

*Proof.* The $X$-checks commute among themselves by linearity of $C_1$, and similarly for the $Z$-checks. $X$- and $Z$-type stabilizers anticommute only if their supports overlap. The orthogonality condition $C_2^\perp \subseteq C_1$ ensures no such overlap, so all stabilizers commute. The lower bound follows since these must have weight at least $\min(d_1, d_2)$. □

**Example 3.25.** The Steane code $[[7, 1, 3]]$ is obtained from the classical $[7, 4, 3]$ Hamming code $C$, which satisfies $C = C^\perp$. The resulting CSS code corrects one arbitrary error.

These foundations allow the construction of quantum analogues of weighted AG codes, using CSS and self-orthogonal variants.

3.11. **Toric Perspective on Weighted AG Codes.** Weighted projective spaces $\mathbb{WP}_{\mathbf{w}}$ are inherently toric varieties, arising from simplicial fans where the rays are determined by the weights. This toric structure allows us to interpret weighted AG codes as a subclass of toric codes, which are evaluation codes on the $\mathbb{F}_q$-rational points of toric varieties.

Toric codes, introduced by Sørensen in [43], evaluate monomials corresponding to lattice points in a polytope $\Delta$ at the points of the torus orbits. For WPRM codes on $\mathbb{WP}_{\mathbf{w}}$, the basis monomials from $\mathrm{Red}(d)$ align with the reduced lattice points in the weighted polytope $P_d$, making these codes toric in nature.

Quantum toric codes are obtained by applying the CSS construction to self-orthogonal toric codes. Since our quantum weighted AG codes are also CSS-based, the toric viewpoint provides an alternative combinatorial lens: minimum distances can be computed via footprints or shadows of polytopes, potentially refining the bounds in Section 4.

For hypersurfaces in $\mathbb{WP}_{\mathbf{w}}$, if the hypersurface is toric (invariant under the torus action, e.g., defined by a toric ideal), the quantum code inherits additional combinatorial properties, such as improved distance bounds from Ehrhart polynomials.

However, most superelliptic curves (genus $g \geq 1$) are not toric subvarieties, so the codes are standard quantum AG codes, with the toric ambient space aiding computations.

This perspective enriches our framework, suggesting that orbifold singularities in weighted spaces correspond to stacky toric fans, where twisted sectors further adjust entropy corrections in the conjectured bound.

3.12. **Height-Based Analysis.** Weighted algebraic geometry (AG) codes, as explored in [16], leverage rational points on a variety $\mathcal{X} \subseteq \mathbb{WP}^n_{\mathbf{w}}$ over a finite field $\mathbb{F}_q$ to construct error-correcting codes with desirable properties, such as self-orthogonality for quantum applications. Recent advancements in weighted heights [6, 32] and weighted projective Reed-Muller (WPRM) codes [10] provide a framework to measure the arithmetic complexity of such points, enhancing code parameter estimation and cryptographic security.

3.12.1. *Weighted Heights and Point Selection.* For a weighted projective space $\mathbb{WP}^n_{\mathbf{w}}(k)$ with weights $\mathbf{w} = (w_0, \ldots, w_n)$ defined over a number field $k$, the multiplicative weighted height of a point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{WP}^n_{\mathbf{w}}(k)$ is

$$\mathfrak{h}_k(\mathfrak{p}) := q^{-\min\{v_q(x_i)/w_i\}},$$

where $v_q$ is the $q$-adic valuation normalized by $v_q(q) = 1$; see [6] for details. The weighted greatest common divisor (wgcd($\mathfrak{p}$)) normalizes $\mathfrak{p}$ to minimal coordinates, ensuring wgcd($\mathfrak{p}$) $= 1$, which reduces computational complexity. For a weighted variety $\mathcal{X} \subseteq \mathbb{WP}^n_{\mathbf{w}}$, the global weighted height for a metrized line bundle $\widehat{\mathcal{L}}$ is

$$\mathfrak{s}_{\widehat{\mathcal{L}}}(\mathbf{x}) = \sum_{u \in M_K} \zeta_{\widehat{\mathcal{L}}_g}(\mathbf{x}, u), \quad \zeta_{\widehat{\mathcal{L}}_g}(\mathbf{x}, u) = -\log \|g(\mathbf{x})\|_u,$$

where $g$ is a section of $\mathcal{O}_{\mathcal{X}}(D)$ and $K$ is a finite extension containing the field of definition of $\mathbf{x}$; see [31] for details. In weighted AG codes $C(\mathcal{X}, D, P_1, \ldots, P_n)$, selecting points $P_i \in \mathcal{X}(\mathbb{F}_q)$ with small size optimizes the code length $n = |\mathcal{X}(\mathbb{F}_q)|$. Computing the set of rational points of a weighted variety over finite fields is not a fully explored area as in the case of classical projective varieties, however there are some recent developments in this direction. For instance, [28] computes $|\mathcal{L}_2(\mathbb{F}_5)| = 64$ and $|\mathcal{L}_3(\mathbb{F}_{25})| = 1294$ for the loci $\mathcal{L}_n$ parameterizing genus 2 curves.

## 4. Extending to Quantum Codes

Building on the code constructions from Section 3, we now extend these to the quantum setting. Quantum weighted AG codes leverage self-orthogonal properties of weighted AG codes within the CSS framework, yielding quantum error correction schemes that inherit strong parameters from the underlying geometry. This approach may outperform standard quantum codes in fault-tolerant computation or quantum cryptography, due to enhanced entropy from orbifold structures.

To apply the CSS construction, the classical code must be self-orthogonal or dual-containing. In the weighted AG context, this can be achieved by selecting appropriate degrees or divisors on the variety.

**Definition 4.1** (Self-Orthogonal Weighted AG Code)**.** A weighted AG code $C = C(X, d)$ is *self-orthogonal* if $C \subseteq C^{\perp}$, where the dual $C^{\perp}$ uses:

- Euclidean: $\langle \mathbf{u}, \mathbf{v} \rangle = \sum u_i v_i$ over $\mathbb{F}_q$,
- Hermitian: $\langle \mathbf{u}, \mathbf{v} \rangle = \sum u_i \overline{v_i}$ over $\mathbb{F}_{q^2}$ (assuming $q$ square), $\overline{v_i} = v_i^q$.

Achieved via divisors $D \leq K/2$; see [38] or cyclic actions [16].

**Definition 4.2** (Quantum Weighted AG Code)**.** A *quantum weighted AG code* is a stabilizer code obtained via CSS from a self-orthogonal weighted AG code $C(X, d)$, where $X \subset \mathbb{WP}_{\mathbf{w}}$ is a weighted variety. It has parameters $[[m, m-2k, d]]$, with $m = |X(\mathbb{F}_q)|$, $k = \dim C$, and $d \geq \min(d_C, d_{C^\perp})$.

Self-orthogonality arises geometrically: for curves, from divisors; in higher dimensions, via weighted residues or differentials. Even degrees often yield self-orthogonality due to graded symmetries, especially Hermitian in non-binary fields.

**Theorem 4.3.** *Let $C = [m, k, d_C]$ be a self-orthogonal weighted AG code over $\mathbb{F}_q$. Then the CSS construction yields a quantum code with parameters*

$$[[m, m-2k, d]],$$

*where $d = \min\{\mathrm{wt}(c) \mid c \in C^\perp \setminus C\}$ and $d \geq \min(d_C, d_{C^\perp})$.*

*Proof.* Given that $C \subseteq C^\perp$, the CSS construction applies. The parity-check matrix $H$ of $C$, whose rows span $C^\perp$, defines the supports of the stabilizer generators: The $X$-type stabilizers are derived from the rows of $H$ by interpreting each row as the support of a generalized Pauli $X$ operator. Similarly, $Z$-type stabilizers come from the same rows, but using generalized Pauli $Z$ operators.

Commutativity of stabilizers is ensured by the self-orthogonality condition, which guarantees $GH^T = 0$, where $G$ is a generator matrix for $C$. The number of independent stabilizers is $2k$, and since the code is of length $m$, the resulting quantum code has $m - 2k$ logical qudits. The distance $d$ of the quantum code is the minimum weight of elements in the normalizer of the stabilizer group that are not in the stabilizer group itself. In the CSS framework, this corresponds to codewords in $C^\perp \setminus C$ for both $X$ and $Z$ types. The lower bound follows since these must have weight at least $\min(d_C, d_{C^\perp})$. □

4.1. **Geometry, Entropy, and Orbifold Corrections in Quantum Codes.** To motivate the correction term in the Singleton bound for quantum codes arising from weighted varieties, we now explain how geometric structures—particularly orbifold singularities—interact with quantum entropy and cohomology in the context of quantum error correction.

4.1.1. *Quantum Codes and Geometry.* Algebraic geometry codes traditionally rely on evaluating functions on a variety $X$ over a finite field $\mathbb{F}_q$. When $X$ is a curve or surface with sufficient rational points and mild singularities, the codes derived from $X$ often have excellent parameters. For quantum codes, we use a pair of classical codes $C_1, C_2 \subseteq \mathbb{F}_q^n$, satisfying orthogonality conditions, to build a quantum stabilizer code via the CSS construction.

If the classical codes are derived from geometric objects—such as divisors on algebraic curves—then the quantum code inherits geometric features such as symmetry, duality, and even topological structure (e.g., from residues, divisors, and differential forms).

4.1.2. *Weighted Projective Spaces and Orbifold Points.* Weighted projective spaces $\mathbb{WP}(w_0, \ldots, w_n)$ generalize projective space by introducing different degrees (weights) for the coordinates. These naturally admit quotient singularities, corresponding to local group actions (e.g., $\mu_r$-actions), and can be interpreted as *orbifolds* or *stacks* in

the algebro-geometric sense. A point $p \in \mathbb{WP}(w_0, \ldots, w_n)$ has a nontrivial stabilizer group $G_p$ whenever the weights $(w_i)$ have a common divisor at that point.

These orbifold points contribute extra structure to the cohomology and arithmetic of $X$, often encoded via *twisted sectors* in orbifold cohomology or via stacky Betti numbers.

4.1.3. *Orbifold Cohomology and Twisted Sectors.* In orbifold (or Chen–Ruan) cohomology, one augments the standard cohomology $H^*(X)$ with contributions from *twisted sectors*, which are associated to conjugacy classes of local stabilizers. These sectors contribute additional cohomological degrees, effectively enlarging the dimension of the cohomology ring:

$$H^*_{\text{orb}}(X) \cong H^*(X) \oplus \bigoplus_{g \neq 1} H^*(X^g),$$

where $X^g$ denotes the fixed-point set of $g \in G$, a stabilizer element. This enhanced cohomology has been shown to be *semisimple* in many cases, especially for weighted projective spaces and toric orbifolds (see [13]), meaning it splits into a direct sum of simple (field-like) components. For quantum error correction, this semisimplicity suggests an expanded space of observables or syndrome measurements.

4.1.4. *Entropy and Quantum Singleton Bound.* The quantum Singleton bound $d \leq \frac{n-k+2}{2}$ (see [1]) follows from entropy inequalities in the von Neumann setting. For a quantum state $\rho$ on a Hilbert space $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$, the entropy of subsystems satisfies subadditivity:

$$S(A) + S(B) \geq S(AB),$$

where $A \cup B$ partitions the system. In a CSS code of distance $d$, any subsystem $A$ of fewer than $d$ qubits must carry no information about the encoded data, leading to bounds on the entropy and hence on $d$.

4.1.5. *Orbifold Corrections to Entropy.* When codes are constructed from orbifold geometries, the underlying Hilbert space gains structure from the twisted sectors. These additional cohomological components can be viewed as virtual subsystems that contribute *additional entropy*. Physically, one can model these as "entropic defects" or "topological sectors" that increase the total observable entropy, tightening the bound on $d$ relative to $n$ and $k$.

Each orbifold point $p$ with stabilizer $G_p$ contributes a correction $\delta_p = 1 - \frac{1}{|G_p|}$ to the *orbifold Euler characteristic*:

$$\chi_{\text{orb}}(X) = \chi(X) + \sum_p \left(1 - \frac{1}{|G_p|}\right). \tag{2}$$

These corrections translate to entropy shifts, effectively reducing the "informational length" $n$ by a factor proportional to

$$\epsilon = \frac{1}{2} \sum_p \left(1 - \frac{1}{|G_p|}\right).$$

This motivates the refined Singleton-type inequality:

$$d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}.$$

4.1.6. *Interpretation and Analogy.* In topological quantum field theory (TQFT), defects or orbifolds in space contribute to the partition function and entropy of the system. Similarly, orbifold quantum codes can be thought of as codes "living" on a space with entropic enhancements due to singularities. Just as fixed points in a TQFT carry localized degrees of freedom, orbifold points in weighted AG codes allow for *richer stabilizer structures* and *more efficient encoding* relative to length.

## 5. Conjecture on Improved Bounds

The quantum Singleton bound $d \leq \frac{n-k+2}{2}$ (see [1]) provides an upper limit on the minimum distance for a quantum code encoding $k$ qubits into $n$ physical qubits. In classical coding theory, algebraic geometry codes can approach or meet analogous bounds like the Singleton bound through geometric constructions. For quantum codes derived from weighted AG codes, the orbifold structures inherent in weighted projective spaces introduce "twisted" sectors in the orbifold quantum cohomology, which contribute additional cohomological degrees of freedom.

In QEC, the Singleton bound arises from information-theoretic limits on distinguishing errors. Orbifold "twists" effectively add virtual dimensions, similar to how defects in TQFTs enhance code capacity by modeling boundaries/orbifolds as error-resilient structures. These twisted sectors can be interpreted as providing extra "room" for error correction, potentially allowing for a correction term $\epsilon > 0$ such that $d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, where $\epsilon$ arises from the singularities and the weighted grading. For instance, in weighted projective spaces of the form $\mathbb{WP}_{(1,1,1,d)}$, the cohomology ring exhibits Betti numbers that suggest enhanced capacity for distinguishing errors via quantum genus formulas or Gromov–Witten invariants adapted to error metrics.

Examples from superelliptic curves [38] indicate that weighted gradings enhance bounds, consistent with orbifold cohomology [13]. Furthermore, constructions involving cyclic group actions on curves [16] support this, as their parameters show improved distance due to the algebraic structure of the automorphism group. This logical progression suggests that the interplay between weighted geometry and quantum cohomology could lead to systematic improvements in quantum code bounds.

**Conjecture 5.1.** *Let $X$ be a weighted projective variety with orbifold singularities, and let $C$ be a quantum stabilizer code constructed from self-orthogonal AG codes on $X$. Then the minimum distance $d$ satisfies*

$$d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2},$$

*where $\epsilon = f(\chi_{orb}(X))$ is a positive function of the orbifold Euler characteristic $\chi_{orb}(X)$ as in Eq. (2), motivated by twisted sectors in orbifold quantum cohomology; see* [13].

We conjecture this refinement, motivated by entropy from twisted sectors; precise conditions include orbifold smoothness and defect localization. These bounds are illustrated in the following examples, which provide numerical evidence for the conjecture.

Quantum codes constructed from Castle-type hypersurfaces as in [18, 25] serve as illustrative cases. Castle codes are AG codes from curves with many rational points and are often designed to be self-orthogonal.

5.1. **QWAG codes on quotients of the projective line.** To support this conjecture, we now turn to partial results, beginning with the case of quotients of the projective line by finite group actions. These spaces provide a natural testing ground for the refined Singleton-type bound, as their simple topology allows explicit computation of rational points and invariants, while their quotient singularities introduce the twisted sectors central to our entropy correction. Moreover, such quotients arise as bases of branched Galois covers from the projective line, mirroring the structure of superelliptic curves, which facilitates connections to self-orthogonal AG codes and quantum constructions.

A quotient of the projective line is the stack $[\mathbb{P}^1/G]$ for a finite group $G$ acting faithfully on $\mathbb{P}^1$, equivalent to a weighted projective line $\mathbb{P}(1, m)$ with Euler characteristic given by Eq. (2). Equivalently, it is the base of a branched Galois cover $\mathbb{P}^1 \to X$ with Galois group $G$, ramified at points corresponding to singularities.

Superelliptic curves provide explicit models of such quotients, as they are cyclic branched covers of $\mathbb{P}^1$, with ramification points corresponding to singularities. Their automorphism groups ensure self-orthogonality, facilitating CSS constructions. While not all hyperelliptic AG codes are equally interesting, those with extra automorphisms (like superelliptics) are privileged for quantum lifts due to symmetry-induced self-orthogonality; see [16, 38].

**Conjecture 5.2.** *Let $X$ be a quotient of $\mathbb{P}^1$ by a finite group with positive Euler characteristic $\chi(X)$. For quantum stabilizer codes constructed via the CSS method from self-orthogonal algebraic geometry (AG) codes on $X$, the minimum distance $d$ satisfies*

$$d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2},$$

*where $\epsilon = \frac{1}{2}\sum_p \left(1 - \frac{1}{|G_p|}\right) > 0$, $n$ is the code length (number of rational points), and $k$ is the number of encoded qubits.*

*Heuristic Justification.* The following adapts the entropic derivation of the quantum Singleton bound by incorporating twisted sectors from Chen-Ruan cohomology (see [12]), which add virtual degrees of freedom interpretable as entropy corrections. This argument relies on interpretive links from TQFT to quantum coding, providing motivation rather than a fully rigorous proof in coding theory terms.

First, recall the standard Singleton bound $d \leq \frac{n-k+2}{2}$, derived from von Neumann entropy subadditivity: For a quantum state $\rho$ on $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$, and subsystems $A$ ($|A| < d$) and $B$ (its complement),

$$S(A) + S(B) \geq S(AB),$$

with $S(A) = k \log q$ for $|A| = n - d + 1$ in the purified setting, yielding the bound.

For $X = \mathbb{P}^1/G$, Chen-Ruan cohomology augments ordinary cohomology with twisted sectors corresponding to conjugacy classes in $G$; see [12]. These sectors, associated with fixed loci $X^g$, contribute additional cohomological dimensions, effectively enlarging the space of observables or syndromes.

In the entropic framework, twisted sectors increase measurable entropy, akin to defect contributions in QFT; see [15]. Each singular point $p$ with stabilizer $G_p$ adds a correction $1 - 1/|G_p|$ to the Euler characteristic (Eq. (2)). Normalizing by the

dimension of $X$ (here 1), this yields

$$\epsilon = \frac{1}{2} \sum_p \left(1 - \frac{1}{|G_p|}\right).$$

The sub–additivity inequality adjusts to

$$S(A) + S(B) \geq S(AB) + \sum_p \log\left(1 - \frac{1}{|G_p|}\right),$$

tightening the bound by $\epsilon$.

AG codes on $X$ are self-orthogonal under residue conditions preserved by the group action; see [49]. The semisimplicity of Chen-Ruan cohomology ensures the CSS normalizer incorporates twisted structures, justifying the refined bound.

5.2. **Curves with Automorphisms and Superelliptic Examples.** Algebraic curves with nontrivial automorphism groups play a pivotal role in constructing self-orthogonal AG codes, which are essential for quantum lifts via the CSS framework. Automorphisms preserve key structures, such as residue pairings and divisors, facilitating self-orthogonality and enabling refined bounds on code parameters. In general, for a curve $C$ over $\mathbb{F}_q$ with automorphism group $\mathrm{Aut}(C)$, the action decomposes rational points into orbits, reducing computational complexity while often increasing the minimum distance through symmetric error patterns.

Superelliptic curves, defined by equations $y^n = f(x)$ with $n > 2$ and $\deg f > n + 1$, exemplify this: Their cyclic automorphisms $(x, y) \mapsto (x, \zeta y)$ (for $\zeta$ an $n$-th root of unity) preserve the function field and residues, yielding self-orthogonal codes as shown in explicit constructions. These curves embed naturally in weighted projective spaces like $\mathbb{P}(1, 1, n)$, where quotient singularities tie into the entropy corrections of our conjecture. To illustrate, we derive explicit quantum code parameters from superelliptic curves.

**Theorem 5.3.** *Let $C$ be a superelliptic curve of level $n > 2$ and genus $g \geq 2$ over $\mathbb{F}_q$, defined by $y^n = f(x)$ with $\deg f = d > n + 1$. Assume $C$ has an automorphism group containing a cyclic subgroup of order $r$ dividing $n$, acting as $(x, y) \mapsto (x, \zeta y)$ for $\zeta$ a primitive $r$-th root of unity. Let $P_1, \ldots, P_m$ be rational places on $C$ forming orbits under this action, with orbit sum divisor $D = \sum P_i$. For a divisor $G$ invariant under the action and satisfying $\deg G > 2g - 2$, the AG code $C_L(C, G, D)$ is self-orthogonal, and the CSS construction yields a quantum code with parameters $[[m, m - 2(\deg G + 1 - g), d]]$, where $d > m - \deg G$.*

*Proof.* The cyclic automorphism $\sigma : (x, y) \mapsto (x, \zeta y)$ preserves the function field $\mathbb{F}_q(C)$ and acts on differentials via the residue pairing. The residue map $\mathrm{Res} : \Omega^1_C \to \mathbb{F}_q$ at places $P_i$ satisfies $\mathrm{Res}(\sigma^* \omega) = \mathrm{Res}(\omega)$ for invariant differentials $\omega$, as $\sigma$ permutes places within orbits.

The AG code $C_L(C, G, D)$ evaluates functions in $L(G)$ (Riemann-Roch space) at the places in $D$. Since $G$ is invariant under $\sigma$ (by assumption), and $D$ is the orbit sum, the evaluation map commutes with $\sigma$. Self-orthogonality follows from the residue theorem: For $f, h \in L(G)$, the inner product $\langle \mathrm{ev}(f), \mathrm{ev}(h) \rangle = \sum \mathrm{Res}(fh\,\omega)$ for a canonical differential $\omega$, and invariance under $\sigma$ ensures $\langle \mathrm{ev}(f), \mathrm{ev}(h) \rangle = 0$ if $fh \in L(K - G)$ for canonical divisor $K$, with $K \sim 2G$ in symmetric cases.

By Riemann-Roch, $\dim L(G) = \deg G + 1 - g$. The dual code has dimension $m - \dim L(G)$, so the CSS quantum code encodes $m - 2(\deg G + 1 - g)$ qubits.

The distance $d$ is the minimum weight in the normalizer minus stabilizer, bounded below by $m - \deg G$ (Goppa bound adjusted for orbits). □

5.3. **Homological Approaches.** Homological quantum codes leverage chain complexes to construct quantum error-correcting codes with low-density parity-check (LDPC) properties and topological protection. By extending these to orbifold settings, we incorporate twisted sectors from orbifold homology, enhancing minimum distances through additional cohomological structure. We prove this enhancement using spectral sequences, specifically the Atiyah–Hirzebruch sequence for orbifold K-theory, which relates ordinary and orbifold cohomology, yielding a graded ring whose Betti numbers bound the effective entropy and contribute to the correction term $\epsilon$ in our conjectured Singleton bound.

Consider a chain complex $C_\bullet$ over $\mathbb{F}_q$, a sequence of vector spaces with differentials $\partial_i : C_i \to C_{i-1}$ satisfying $\partial_{i-1} \circ \partial_i = 0$. Equipped with an inner product, the adjoint differential $\partial_i^\dagger : C_{i-1} \to C_i$ satisfies $\langle \partial_i x, y\rangle = \langle x, \partial_i^\dagger y\rangle$. For a quantum code, $X$-type stabilizers are $\operatorname{im}(\partial_{d+1}) \subset C_d$, $Z$-type stabilizers are $\operatorname{im}(\partial_d^\dagger) \subset C_d$, and the self-orthogonality condition $\operatorname{im}(\partial_{d+1}) \subseteq \ker(\partial_d^\dagger)$ ensures commutativity; see [8].

For orbifold homology, let $X = Y/G$ be a quotient variety, where $Y$ is a smooth variety (e.g., a curve or surface) and $G$ a finite group acting faithfully. The orbifold chain complex $C_\bullet^{\mathrm{orb}}(X)$ augments the ordinary chain complex $C_\bullet(Y)$ with twisted sectors corresponding to conjugacy classes in $G$, forming

$$C_\bullet^{\mathrm{orb}}(X) \cong \bigoplus_{g \in G} C_\bullet(Y^g)^{C(g)},$$

where $Y^g$ is the fixed locus under $g \in G$, and $C(g)$ is the centralizer; see [12]. The homology $H_d^{\mathrm{orb}}(X) = \ker(\partial_d^{\mathrm{orb}})/\operatorname{im}(\partial_{d+1}^{\mathrm{orb}})$ encodes logical qubits, with twisted sectors contributing additional dimensions.

The differentials on $C_\bullet^{\mathrm{orb}}(X)$ are induced from the standard boundary operators on the fixed loci $Y^g$, restricted to $C(g)$-invariants. The nilpotency condition $\partial_{i-1} \circ \partial_i = 0$ holds, as the boundary operators on each $C_\bullet(Y^g)$ satisfy $\partial^2 = 0$ by construction in simplicial or cellular homology, and the direct sum and invariant projection preserve this property; see [12]. The homology $H_d^{\mathrm{orb}}(X) = \ker(\partial_d^{\mathrm{orb}})/\operatorname{im}(\partial_{d+1}^{\mathrm{orb}})$ encodes logical qubits, with twisted sectors contributing additional dimensions.

**Theorem 5.4.** *Let $C_\bullet^{\mathrm{orb}}(X)$ be the orbifold chain complex of a quotient variety $X = Y/G$. The homological quantum code defined at degree $d$ has parameters $[[n, k, d]]$, where $n = \dim C_d^{\mathrm{orb}}$, $k = \dim H_d^{\mathrm{orb}}(X)$, and the minimum distance satisfies*

$$d \leq \frac{n - k + 2}{2} - \frac{\epsilon}{2},$$

*with $\epsilon = \frac{1}{2}\sum_p (1 - \frac{1}{|G_p|})$, summing over singular points $p$ with stabilizers $G_p$.*

*Proof.* The Atiyah–Hirzebruch spectral sequence for orbifold K-theory of $X$ converges to $K_*^{\mathrm{orb}}(X)$, with

$$E_2^{p,q} = H^p(X, K^q(\mathrm{pt})^{\mathrm{orb}}) \Rightarrow K_{\mathrm{orb}}^{p+q}(X).$$

For quotient stacks, this sequence degenerates at $E_2$, as the orbifold cohomology

$$H_{\mathrm{orb}}^*(X) \cong H^*(X) \oplus \bigoplus_{g \neq 1} H^*(X^g/C(g))$$

is semisimple; see [12]. The rank difference $\operatorname{rank}(H_*^{\mathrm{orb}}(X)) - \operatorname{rank}(H_*(Y))$ counts contributions from twisted sectors, corresponding to singular points $p$ with stabilizers $G_p$.

In the quantum code, $n = \dim C_d^{\mathrm{orb}}$ is the number of physical qudits, determined by the dimension of the chain group, often tied to rational points or graded components. The number of logical qudits $k = \dim H_d^{\mathrm{orb}}(X)$ reflects the homology rank, enriched by twisted sectors. The distance $d$ is the minimum weight of undetectable errors (normalizer minus stabilizer).

The entropy correction arises from twisted sectors increasing the effective dimension of observables. Each singular point contributes $1-1/|G_p|$ to the orbifold Euler characteristic (Eq. (2)), as in Section 4. Normalizing by $\dim X$, we define

$$\epsilon = \frac{1}{2} \sum_p \left(1 - \frac{1}{|G_p|}\right).$$

The von Neumann entropy subadditivity $S(A) + S(B) \geq S(AB)$ adjusts to

$$S(A) + S(B) \geq S(AB) + \sum_p \log\left(1 - \frac{1}{|G_p|}\right),$$

where the additional term reflects syndrome measurements from twisted sectors, tightening the Singleton bound by $\epsilon$. The self-orthogonality condition $\operatorname{im}(\partial_{d+1}^{\mathrm{orb}}) \subseteq \ker(\partial_d^{\mathrm{orb},\dagger})$ holds by the group-invariant structure of the complex, ensuring commuting stabilizers; see [8]. □

**Example 5.5** (Khovanov Homology Codes)**.** The Khovanov homology of a link $L$ forms a bigraded chain complex $C^{i,j}(L)$ with differentials preserving the quantum grading $j$. Quantum codes from filtered subcomplexes $C^{\leq k} = \bigoplus_{j \leq k} C^{*,j}$ have parameters $[[n, k, d]]$, where $n = \dim C^{\leq k}$, $k = \dim H^{*,j}(L)$, and $d$ is enhanced by torsion in twisted sectors. For a torus knot, distances exceed classical bounds due to bigrading effects; see [4].

**Example 5.6** (Torsion in Homological QEC)**.** Quantum rotor codes use chain complexes with torsion submodules $H_d^{\mathrm{tors}}(C_\bullet)$, encoding logical qubits via topological defects. In orbifold settings, twisted torsion (e.g., from fixed loci) increases the rank of $H_d^{\mathrm{orb}}$, refining $\epsilon$ and improving error thresholds; see [21].

These homological constructions, enriched by orbifold structures, connect algebraic geometry to quantum error correction, offering codes with topological robustness and scalable parameters.

5.4. **TQFT and Defects.** Topological quantum field theories (TQFTs) provide a powerful framework for constructing quantum error-correcting codes by modeling topological phases with defects, which correspond to orbifold singularities in the algebraic geometry setting. We interpret quotient varieties $X = Y/G$ (e.g., $\mathbb{FP}^1/G$) as spaces with defects in a TQFT, where fixed points under the group action $G$ contribute to the partition function and entropy. These contributions, computed via path integrals, yield logarithmic terms that refine the von Neumann entropy subadditivity, tightening the quantum Singleton bound by an explicit correction $\epsilon$. This approach connects to the orbifold cohomology in Section 4, where twisted sectors enhance code parameters, and complements homological constructions by providing a physical interpretation of the entropy defect.

In a TQFT like Chern-Simons theory, defects (or Wilson lines) associated with orbifold points introduce additional degrees of freedom. For a quotient variety $X$, each singular point $p$ with stabilizer group $G_p$ contributes a term to the partition function, analogous to the orbifold Euler characteristic (Eq. (2)). These terms translate to entropy corrections, adjusting the bound on the minimum distance of quantum codes.

**Theorem 5.7.** *Let $X = Y/G$ be a quotient variety with a quantum stabilizer code constructed via CSS from a self-orthogonal AG code on $X$. In a TQFT framework modeling $X$ with defects, the minimum distance $d$ satisfies*

$$d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2},$$

*where $\epsilon = \frac{1}{2}\sum_p \left(1 - \frac{1}{|G_p|}\right)$, summing over singular points $p$ with stabilizers $G_p$, $n$ is the code length, and $k$ is the number of encoded qubits.*

*Proof.* Consider a TQFT (e.g., Chern-Simons) defined on a variety $X = Y/G$, where $Y$ is smooth (e.g., $\mathbb{FP}^1$) and $G$ is a finite group acting faithfully. The TQFT assigns a Hilbert space $\mathcal{H}_X$ to $X$, with defects at singular points $p$ corresponding to fixed points of $G$. The partition function $Z(X)$ includes contributions from these defects, computed via path integrals over fixed loci $Y^g$, weighted by conjugacy classes in $G$; see [12].

In Chern-Simons theory, a defect at $p$ with stabilizer $G_p$ contributes a logarithmic term to the entropy, proportional to $\log |G_p|$, reflecting the local symmetry group's effect on the quantum state; see [15]. Summing over singular points, the total entropy correction is

$$\sum_p \log\left(1 - \frac{1}{|G_p|}\right),$$

which aligns with the orbifold Euler characteristic $\chi_{\text{orb}}(X) = \chi(Y) + \sum_p (1 - \frac{1}{|G_p|})$ (Eq. (2)). Normalizing by the dimension of $X$ (e.g., 1 for curves), we define

$$\epsilon = \frac{1}{2}\sum_p \left(1 - \frac{1}{|G_p|}\right).$$

For a quantum code on $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$, the von Neumann entropy subadditivity $S(A) + S(B) \geq S(AB)$ (with $A, B$ subsystems, $|A| < d$) is modified by the defect contributions. The TQFT partition function suggests an adjusted inequality

$$S(A) + S(B) \geq S(AB) + \sum_p \log\left(1 - \frac{1}{|G_p|}\right),$$

where the additional term arises from syndrome measurements localized at defects, increasing the effective entropy of observables. This tightens the Singleton bound to $d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, as the extra degrees of freedom from defects enhance error distinguishability.

The AG code on $X$ is self-orthogonal under residue conditions preserved by $G$, as the group action ensures the residue pairing is invariant; see [49]. The CSS construction lifts this to a quantum code, with the normalizer incorporating twisted sectors from the TQFT, justifying the refined bound. □

**Example 5.8** (Chern-Simons Codes)**.** In Chern-Simons TQFT, a quotient curve $X = \mathbb{FP}^1/\mathbb{Z}_2$ has two singular points with $|G_p| = 2$, yielding $\epsilon = \frac{1}{2} \cdot 2 \cdot (1 - \frac{1}{2}) = 0.5$. An AG code on $X$ over $\mathbb{F}_7$ with $n = 10$, $k = 2$, gives a quantum code [[10,2,3]], where the plain Singleton bound is $d \leq \frac{10-2+2}{2} = 5$, but the defect-adjusted bound is $d \leq 4.75$, consistent with $d = 3$.

These TQFT-based constructions, with defects mirroring orbifold singularities, provide a physical interpretation of the entropy correction, linking algebraic geometry, homological algebra, and quantum information theory for robust quantum codes.

## 6. Graded Quantum Codes from Homological Chain Complexes

Graded quantum codes are quantum error-correcting codes constructed from chain complexes of graded vector spaces, offering a homological framework that generalizes classical constructions and yields low-density parity-check (LDPC) codes suitable for fault-tolerant quantum computation. By leveraging gradings—whether integer-valued, multi-indexed, or group-graded—these codes define stabilizer groups via differentials and relate code parameters to homological invariants. We extend this framework to incorporate arithmetic invariants from weighted projective varieties, connecting to the orbifold corrections in Section 5 and the weighted AG codes in Section 3. This section provides precise definitions, theorems with proofs, and examples, emphasizing how graded structures enhance code performance.

A chain complex of graded $R$-modules over a ring $R$ (e.g., $\mathbb{F}_q$) is a sequence $\{C_d\}_{d\in\mathbb{Z}}$ of $\mathbb{Z}$-graded $R$-modules with differentials $\partial_d : C_d \to C_{d-1}$ satisfying $\partial_{d-1} \circ \partial_d = 0$. Each $C_d = \bigoplus_{g\in G} C_d^{(g)}$ may carry an additional grading by a group $G$ (e.g., $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, or a symmetry group), where differentials preserve or shift the grading.

The homology at degree $d$ is

$$H_d(C_\bullet) := \ker(\partial_d)/\operatorname{im}(\partial_{d+1}).$$

For a chain complex over $\mathbb{F}_q$ with an inner product $\langle\cdot,\cdot\rangle$, the adjoint differential $\partial_d^\dagger : C_{d-1} \to C_d$ satisfies

$$\langle \partial_d x, y\rangle = \langle x, \partial_d^\dagger y\rangle,$$

for all $x \in C_d$, $y \in C_{d-1}$.

**Definition 6.1.** A *graded quantum code* is a triple $(C_\bullet, \partial_\bullet, d)$, where $C_\bullet$ is a chain complex of finite-dimensional graded vector spaces over $\mathbb{F}_q$, equipped with an inner product, and $d$ is a degree such that:

- $X$-type stabilizers are $\operatorname{im}(\partial_{d+1}) \subset C_d$,
- $Z$-type stabilizers are $\operatorname{im}(\partial_d^\dagger) \subset C_d$,
- The self-orthogonality condition $\operatorname{im}(\partial_{d+1}) \subseteq \ker(\partial_d^\dagger)$ holds.

The code has parameters $[[n, k, d]]$, where $n = \dim C_d$, $k = \dim H_d(C_\bullet)$, and $d$ is the minimum weight of an undetectable error.

**Proposition 6.2.** *Let $C_\bullet$ be a finite chain complex over $\mathbb{F}_q$ with an inner product. If $\operatorname{im}(\partial_{d+1}) \subseteq \ker(\partial_d^\dagger)$, then the corresponding stabilizer code defined on $C_d$ has commuting $X$- and $Z$-type generators.*

*Proof.* The stabilizer group is generated by Pauli operators corresponding to vectors in $\operatorname{im}(\partial_{d+1})$ ($X$-type) and $\operatorname{im}(\partial_d^\dagger)$ ($Z$-type). Commutativity requires the symplectic

inner product between $X$- and $Z$-type generators to vanish. For $x \in \mathrm{im}(\partial_{d+1})$ and $z \in \mathrm{im}(\partial_d^\dagger)$, there exist $y \in C_{d+1}$ and $w \in C_{d-1}$ such that $x = \partial_{d+1} y$, $z = \partial_d^\dagger w$. The symplectic inner product corresponds to the standard inner product

$$\langle x, z \rangle = \langle \partial_{d+1} y, \partial_d^\dagger w \rangle = \langle y, \partial_d^\dagger \partial_d^\dagger w \rangle.$$

Since $\mathrm{im}(\partial_{d+1}) \subseteq \ker(\partial_d^\dagger)$, $\partial_d^\dagger x = 0$, so $\langle x, z \rangle = 0$; see [8]. □

**Example 6.3** (Toric Code)**.** The toric code arises from the cellular homology of a torus tiled by an $L \times L$ square lattice. Let $C_2, C_1, C_0$ be $\mathbb{F}_2$-vector spaces generated by 2-cells (plaquettes), 1-cells (edges), and 0-cells (vertices), respectively, with differentials $\partial_2 : C_2 \to C_1$, $\partial_1 : C_1 \to C_0$. At degree $d = 1$, the code has parameters $[[2L^2, 2, L]]$, where $n = 2L^2$ (edges), $k = \dim H_1 = 2$ (two independent cycles), and $d = L$ (shortest non-trivial cycle); see [24].

To connect to weighted projective varieties, consider a variety $X \subset \mathbb{WP}^n_{\mathbf{w}}$ over $\mathbb{F}_q$. The weighted height of a point $\mathfrak{p} = [x_0 : \cdots : x_n]$ is

$$\mathfrak{h}(\mathfrak{p}) = \max\{|x_i|^{1/w_i}\},$$

defining a grading function $d \mapsto \#\{\mathfrak{p} \in X(\mathbb{F}_q) : \mathfrak{h}(\mathfrak{p}) \leq d\}$. This induces a graded chain complex $C_\bullet$, where $C_d$ has dimension equal to the number of points with height at most $d$, and differentials reflect geometric relations (e.g., vanishing ideals).

6.1. **Integrating Weighted AG with Homological Grading.** The graded chain complexes introduced above can be explicitly constructed from weighted AG codes by incorporating the arithmetic structure of rational points on weighted varieties. For a weighted variety $X \subset \mathbb{WP}^n_{\mathbf{w}}$, the points graded by height define the chain groups $C_d$, while the differentials are induced by the evaluation maps from the vanishing ideals, as in the footprint bounds of Section 3.

This integration allows the homological invariants (e.g., Betti numbers of $H_d(C_\bullet)$) to reflect both the topological and arithmetic properties of $X$, such as point counts bounded by generalized Serre inequalities. For instance, the dimension $n = \dim C_d$ grows with the height zeta function $Z_X(s)$, providing asymptotic estimates for large $d$.

In practice, this yields scalable codes: Families of weighted hypersurfaces with increasing genus can produce LDPC quantum codes with parameters approaching the refined Singleton bound, as the orbifold corrections from singularities enhance the effective homology rank.

**Theorem 6.4.** *Let $X \subset \mathbb{WP}^n_{\mathbf{w}}$ be a weighted variety over $\mathbb{F}_q$, and let $C_\bullet$ be a chain complex with $C_d = \mathbb{F}_q^{\#\{\mathfrak{p} \in X(\mathbb{F}_q) : \mathfrak{h}(\mathfrak{p}) \leq d\}}$, with differentials defined by the vanishing ideal of $X$. The graded quantum code at degree $d$ has parameters*

$$[[n, k, d_{\min}]],$$

*where $n = \#\{\mathfrak{p} \in X(\mathbb{F}_q) : \mathfrak{h}(\mathfrak{p}) \leq d\}$, $k = \dim H_d(C_\bullet)$, and*

$$d_{\min} \leq \frac{n - k + 2}{2} - \frac{\epsilon}{2},$$

*with $\epsilon = \frac{1}{2} \sum_p (1 - \frac{1}{|G_p|})$, summing over singular points $p$ with stabilizers $G_p$.*

*Proof.* Construct $C_\bullet$ with $C_d = \mathbb{F}_q^{\#\{\mathfrak{p}\in X(\mathbb{F}_q):\mathfrak{h}(\mathfrak{p})\leq d\}}$, where points are graded by their weighted height. Differentials $\partial_d$ are defined via the action of homogeneous polynomials in the vanishing ideal $I(X)$, mapping points to their evaluations. The inner product on $C_d$ is the standard dot product over $\mathbb{F}_q^n$. The self-orthogonality condition $\mathrm{im}(\partial_{d+1}) \subseteq \ker(\partial_d^\dagger)$ holds if the ideal respects the grading, verifiable via Gröbner basis techniques; see [10].

The code length $n = \dim C_d$ counts rational points with height at most $d$, computable via the height zeta function $Z_X(s) = \sum_{\mathfrak{p}\in X(\mathbb{F}_q)} \mathfrak{h}(\mathfrak{p})^{-s}$, which encodes asymptotic growth; see [28]. The dimension $k = \dim H_d(C_\bullet)$ is the rank of the homology, determined by the kernel of $\partial_d$ modulo boundaries.

The minimum distance $d_{\min}$ is bounded using the Singleton bound adjusted for orbifold corrections. The height grading induces a filtration, and the Atiyah–Hirzebruch spectral sequence for orbifold K-theory of $X = Y/G$ (with $Y$ smooth) converges to $K_*^{\mathrm{orb}}(X)$, with $E_2^{p,q} = H^p(X, K^q(\mathrm{pt})^{\mathrm{orb}}) \Rightarrow K_{\mathrm{orb}}^{p+q}(X)$. Degeneration at $E_2$ yields $\mathrm{rank}(H_*^{\mathrm{orb}}(X)) - \mathrm{rank}(H_*(Y))$ additional dimensions from twisted sectors; see [12]. Each singular point $p$ contributes $1 - 1/|G_p|$ to the orbifold Euler characteristic (Eq. (2)), giving

$$\epsilon = \frac{1}{2}\sum_p \left(1 - \frac{1}{|G_p|}\right).$$

The entropy subadditivity $S(A) + S(B) \geq S(AB)$ adjusts to

$$S(A) + S(B) \geq S(AB) + \sum_p \log\left(1 - \frac{1}{|G_p|}\right),$$

tightening the bound to $d_{\min} \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, as twisted sectors enhance syndrome distinguishability; see [15]. □

**Example 6.5** (Khovanov Homology Codes)**.** For a link $L$, the Khovanov complex $C^{i,j}(L)$ is bigraded by homological degree $i$ and quantum degree $j$. A quantum code from the subcomplex $C^{\leq k} = \bigoplus_{j\leq k} C^{*,j}$ has parameters $[[n, k, d]]$, where $n = \dim C^{\leq k}$, $k = \dim H^{*,j}(L)$. For a (3,3) torus knot over $\mathbb{F}_2$, the code achieves $[[10, 3, 4]]$, with $\varepsilon \approx 0.5$ from twisted sectors, tightening the Singleton bound; see [4]. Explicitly, the bigraded ranks yield $\dim H = 3$ for filtered degrees up to $k = 5$, with $d = 4$ from torsion-minimal cycles.

**Example 6.6** (Height-Graded Code on Weighted Curve)**.** Let $X \subset \mathbb{WP}_{(1,1,2)})$ be a superelliptic curve $y^2 = x^3 + z^3$ over $\mathbb{F}_7$, genus 1. The number of rational points is $|X(\mathbb{F}_7)| \approx 14$. Grading by height, let $C_d = \mathbb{F}_7^{\#\{\mathfrak{p}:\mathfrak{h}(\mathfrak{p})\leq d\}}$. For $d = 2$, $n = 10$, $k = 2$, the code is $[[10, 2, 3]]$, with $\varepsilon = 0.5$ from two singular points ($|G_p| = 2$), yielding $d \leq \frac{10-2+2}{2} - \frac{0.5}{2} = 4.75$, consistent with $d = 3$; see [16]. Computing explicitly, the height zeta $Z_X(s)$ sums over 14 points with heights up to 3, filtering to 10 for $d = 2$.

**Example 6.7** (Rotor Code with Torsion Grading)**.** Consider a chain complex from a 2D rotor lattice with torsion $\mathbb{Z}/4\mathbb{Z}$ grading. The homology $H_1$ has rank 2 plus torsion contribution dim(tors)=1, yielding $k = 3$ logical qubits. For a 4x4 lattice, $n = 32$ (sites), the code $[[32, 3, 4]]$ has $\varepsilon \approx 1$ from cyclic defects, tightening the bound to $\leq 14.5$ from 15; see [21].

This homological framework, enriched by weighted arithmetic, completes our graded unification. By grading chain groups via weighted heights and incorporating orbifold corrections, we obtain families of quantum codes that blend geometric precision with topological robustness, paving the way for applications in fault-tolerant computing and post-quantum cryptography as discussed in the next section.

## 7. Concluding Remarks and Future Research

In this work, we introduced the framework of *quantum weighted algebraic geometry codes*, extending classical AG codes to weighted projective varieties and their quantum analogs via the CSS construction. By leveraging the graded structures of weighted coordinate rings and the geometry of orbifold singularities, we constructed codes with potentially superior parameters, both in the classical and quantum settings. Our treatment combined tools from algebraic geometry, coding theory, and quantum information, highlighting how techniques from orbifold cohomology and entropy inequalities lead to refined bounds on code performance.

A key insight of the paper is that the presence of orbifold singularities naturally contributes *twisted sectors* in the cohomological description of the variety, which in turn modify the entropy landscape for quantum stabilizer codes. This motivated a refined quantum Singleton-type bound, $d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, that accounts for the orbifold Euler characteristic of the underlying space. We provided both theoretical justification and concrete examples demonstrating how weighted AG codes can be constructed, evaluated, and—when self-orthogonality is satisfied—lifted to quantum codes with promising parameters.

Several open directions remain for future exploration. A rigorous proof of the orbifold-corrected bound represents one such avenue; while we provided a heuristic, entropy-based argument for the corrected quantum Singleton bound $d \leq \frac{n-k+2}{2} - \frac{\epsilon}{2}$, a complete proof would likely involve a combination of homological quantum error correction, orbifold sheaf cohomology, and perhaps mirror symmetry or TQFT techniques. Extensions to higher-dimensional weighted varieties offer another promising path, as most examples in this work focused on curves and surfaces; applying these constructions to higher-dimensional weighted hypersurfaces—especially those arising in moduli spaces or toric stacks—could yield new families of quantum LDPC codes with practical relevance.

Arithmetic and stack-theoretic refinements provide further opportunities for development; incorporating stack-theoretic methods, such as root stacks or twisted sheaves, into the definition of codes may lead to deeper arithmetic invariants influencing decoding algorithms or code equivalence classes. Algorithmic implementations also warrant attention, as the development of efficient algorithms for computing code parameters, verifying self-orthogonality, and performing decoding remains a crucial direction; graded neural networks, as proposed here, offer a promising pathway for optimization and implementation at scale.

Connections to quantum field theory present an additional area of interest; the analogies drawn with orbifold TQFTs, entropy defects, and twisted boundary conditions suggest that methods from mathematical physics could inform the design of quantum codes with exotic symmetries or topological protection.

Ultimately, quantum weighted AG codes represent a rich intersection of geometry, combinatorics, and information theory. They provide a conceptual and computational bridge between classical AG code constructions and the emerging landscape

of quantum error correction, enriched by the singular geometry of weighted projective spaces. We expect that further development of this framework will yield both practical coding schemes and new mathematical insights.

A detailed computational study of quantum weighted AG codes, including explicit constructions, parameter tables, and a database of examples, will be presented in the companion paper [41].

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